Asymptotic Stability in a Controlled Stochastic Lotka-Volterra Model with Lévy Noise

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Abstract: We consider a stochastic Lotka-Volterra model for one-predator-two- prey with driven by noise and Lévy jumps. The objective of the paper is to study this Lotka-Volterra model introducing controls on the deterministic part and on the Lévy noise, by means of Lyapunov approaches functions techniques. Assuming linear growth and Lipschitz conditions in the drift and diffusion terms, we prove the boundedness and the exponential stability of its solutions.

Keywords: Lotka-Volterra model, stochastic optimal control, stability, Lévy jumps

1. Introduction

In mathematical ecology, the Lotka-Volterra systems represent one of the most important models to describe population dynamics, because they describe very well many aspects of interactions between species predator-prey in competition, such as persistence, extinction, limit cycles and stability of its solutions [1–3]. When it is introduced control functions in this model, we arrive to study most completely the ecosystem and we can steer the system from an initial configuration to a final configuration. These models are more realistic if we consider natural random environmental variations, considering some stochastic process, as Wiener processes or even better, Lévy jumps, which model aleatory environmental fluctuations in the nature such that hazardous waste pollution or cyclonic storms, by instance. Lévy processes are stochastic processes with stationary and independent increments, like sub-martingales or Markov processes, that is, they are processes $\xi(t)$ such that $\xi(t+s)-\xi(s)$ and $\xi(r)$ are independent distributions with the same probability, for $s, t \geq 0$ and $0 \leq r \leq t$ and they can be thought of as random walks in continuous time [4]. Many papers have studied stochastic Lotka-Volterra models with jumps, analyzing persistence, extinction, boundedness, local stability and more properties [5–7]. The Lotka-Volterra equations have also been applied to laser physics (optical and photonic devices), to describe population inversion and the number of emitted photons, as in [8]. In this case, the authors study their regions of stability and the transformation of a fixed point into a limit cycle. In Ref. [9], the authors consider a competitive Lotka-Volterra population dynamics with jumps without control functions and they investigate the sample Lyapunov exponent for each component of the solution and uniform boundedness of the p–th moment with $p > 0$. Also, in Ref. [10] was studied the asymptotic convergence of a general stochastic population dynamics of the type Lotka-Volterra and driven by Lévy noise, given some important asymptotic path-wise estimation assuming different conditions over the Poisson's process coefficient, but they don't consider any control functions in the processes. In Ref. [11], the authors find conditions under which the solutions to the stochastic differential equations driven by Lévy noise are moment exponentially stable, without any control functions in the processes. Our main results in this paper are: the boundedness of solutions of the stochastic model (Theorem 2) and the exponential stability of the solutions of the system (1), around the static solution (Theorem 3).

2. Problem Formulation

The model to consider here is a controlled jump diffusion process given by the following non-linear stochastic ordinary differential equations system with initial and final conditions:

$$dx = f\big(t, x(t), u(t)\big)dt + g\big(t, x(t), u(t)\big)dW(t) + x(t)u(t)\int_{\mathbb{R}^3} \gamma(t, x(-), z)\tilde{N}(dt, dz), \tag{1}$$

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \quad x_3(0) = x_{30},$$
$$x_1(T) = x_{11}, \quad x_2(T) = x_{21}, \quad x_3(T) = x_{31}, \tag{2}$$

where $f : [0, T] \times R^3 \times R^3 \to R^3$, given by

$$f(t, x, u) = (f^1(t, x, u), f^2(t, x, u), f^3(t, x, u))^T, \tag{3}$$

is a measurable function called the drift, the process $u : R \to R^3$,

$$u(t) = (u_1(t), u_2(t), u_3(t)), \tag{4}$$

is a measurable and bounded function called the control, which belongs to a region control $U \in R^3$ and it is an adapted and cadlag function (continuous on the right and limit on the left), and g(t, x, u), a measurable function defined also on $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$ and $\mathbb{R}^{3\times3}$-valued (3 × 3 - real matrix),

$$g(t, x, u) = (g^1(t, x, u), g^2(t, x, u), g^3(t, x, u)), \tag{5}$$

where

$$g^j(t, x, u) = (g^{1j}(t, x, u), g^{2j}(t, x, u), g^{3j}(t, x, u))^T, 1 \le j \le 3, \tag{6}$$

called the diffusion coefficient and for the compensated Poisson random measure $\tilde{N}(dt, dz)$, we write, according to Lévy decomposition theorem, [12],

$$\tilde{N}(dt, dz) = (\tilde{N}_1(dt, dz), \tilde{N}_2(dt, dz), \tilde{N}_3(dt, dz)), \tag{7}$$

and

$$\tilde{N}_j(dt, dz) = \tilde{N}_j(dt, dz) - \nu_j(dz_j)dt, 1 \le j \le 3, \tag{8}$$

with $N_j(dt, dz)$ Poisson counting measure and $x_i(t-)$ denotes the left hand limit of x at time t. Specifically, we consider the following functions $f(t, x(t), u(t)), g(t, x(t), u(t))$:

$$f(t,x,h)=\begin{pmatrix} x_1(t)\,-\,\beta\, x_1(t)x_2(t)\,-\,\delta x_1(t)x_3(t)\,-\,A_1x_1(t)u_1(t) \\ x_2(t)\,-\,\beta\, x_2(t)x_1(t)\,-\,\epsilon x_2(t)x_3(t)\,-\,A_2x_2(t)u_2(t) \\ -x_3(t)\,-\,\delta\, x_3(t)x_1(t)\,-\,\epsilon x_3(t)x_2(t)\,-\,A_3x_3(t)u_3(t) \end{pmatrix}, \tag{9}$$

$$g(t,x,u)=\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}. \tag{10}$$

where η, ω, κ are positive constants in (0, 1], being the intrinsic growth rates of two preys and predator population, respectively, β, δ, η, and ϵ in (0, 1], are positive constants meaning the contact rates per unit of time between prey-prey, predator-first prey and predator-second prey, respectively. $u_1(t), u_2(t), u_3(t)$ are the controls, representing, by example by the hunting in each population, for which we have modulated their effect with constants $A_1, A_2, A_3 \in (0,1]$. To take into account environmental fluctuations on the prey and the predator populations we introduce standard independents Wiener processes $W_1(t), W_2(t), W_3W(t)$ with parameters $\alpha_1, \alpha_2, \alpha_3 \in (0,1]$, respectively, in three independent random variations for each population, defined over a probability space $(\Omega, \mathcal{F}, P)$ and, finally, $N(t)$ is a Poisson process independent of $W(t)$. In the above, as is conventional, P denotes a probability measure in the sample space Ω of the stochastic process X : [0, T] × Ω → [0, +∞) and E[X] denotes the expected value with respect to the probability measure P, that is, the integral $E[X_T] = \int_\Omega X_T(\omega)dP(\omega)$ in the sense of Lebesgue integration. $\mathcal{F}_s$ denotes the σ-algebra generated by all random variables X_i with $i \leqslant s$; the collection of such σ-algebras forms a filter of the probability space. The class of admissible controls U is the set of F_s-predictable processes with values in U.

Our study will permit to develop techniques to control random variations of some ecosystems and sudden changes in the environment, like Lyapunov approach or geometric techniques of Control Theory [13], as part of an optimal and general strategy for the preservation of species and the harvest of any renewable resource as some animals or plants.

Considering the stochastic differential system (1), in order to guarantee the existence and uniqueness of solutions, we assume the following hypothesis related with the Lipschitz and linear growth conditions in the x variable, for $f(x,t,u), g(x,t,u)$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ the jump coefficient or Poisson's process coefficient, defined by

$$h(x,t,u) \;=\; \int_{R^3} \gamma\,(t,x,z)\, \widetilde{N}(dt,dz). \tag{11}$$

(H1) There exist constants $\kappa_1 < \infty$ and $\kappa_2 < \infty$ such that $f(x,t,u), g(x,t,u)$ and $h(x,t,u)$ satisfy:

a) At most linear growth condition:

$$\begin{gathered} ||\, f(x,t,u)\, ||^{\,2} \leq\; \kappa_1\,(1\,+\,||\,x\,||^2\,), \\ ||\, g(x,t,u)\, ||^2 \;\leq\; \kappa_1\,(1\,+\,||\,x\,||^{\,2}), \\ \int_{R^3} ||\gamma\,(t,x,z)\,||^{\,2}\, \widetilde{N}(dt,dz) \;\leq\; \kappa_1\,(1\,+\,||\,x\,||^2\,). \end{gathered} \tag{12}$$

b) Lipschitz continuity:

$$\begin{gathered} ||\, f(x,t,u)\,-\,f(y,t,u)\,||^{\,2} \leq\; \kappa_2\,||\,x\,-\,y\,||^2\,, \\ ||\, g(x,t,u)\,-\,g(y,t,u)\,||^2 \;\leq\; \kappa_2\,||\,x\,-\,y\,||^2\,, \\ \int_{R^3} ||\gamma\,(t,x,z)\,-\,\gamma\,(t,y,z)\,||^2\, \widetilde{N}(dt,dz) \;\leq\; \kappa_2\,||\,x\,-\,y\,||^2\,. \end{gathered} \tag{13}$$

(H2) Controls are bounded: there exists $\kappa_3 < \infty$, such that $\forall t \in R$:

$$|| \, u(t) \, || \leq \kappa_3 \, . \tag{14}$$

(H3) Jump is locally bounded: for all bounded sets $M \subset R^3$:

$$sup_{x \in M} \, sup_{0 \leq |z| \leq c} \, |\gamma \, (s, x(s-), z) \, | < \infty. \tag{15}$$

Assumptions (H1) and (H2) are reasonable in this model, since the controls represent population hunting, which cannot be excessive, and the size of the jump must also be limited. Finally, we consider the solution of the static problem corresponding to system (1), called $\tilde{x}(t)$. We claim the boundedness of solutions of the stochastic model and the exponential stability of the solutions of the system (1), around the static solution $\tilde{x}(t)$.

3. Exponential Stability

As usual, we introduce a Lyapunov function, which is a Ito-Lévy process $V\,(x,t) \in C^{2,1}\,(\mathbb{R}^3 \times \mathbb{R}^+\,;\,\mathbb{R}^+)$, where $C^{2,1}\,(\mathbb{R}^3 \times \mathbb{R}^+\,;\,\mathbb{R}^+)$, is the family of all non-negative functions $V\,(x,t)$, continuously twice differentiable in x and once in t, defined on $\mathbb{R}^3 \times \mathbb{R}^+$, which guarantees the stability of the solution of the general stochastic differential Eq. (1), and also we introduce the linear operator or diffusion operator $L: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, acting on $V\,(x,t)$, defined as following:

$$LV(t,x) = \frac{\partial V}{\partial t}(t,x) + \langle \frac{\partial V}{\partial x}(t,x), f(t,x,u) \rangle$$

$$+ \frac{1}{2} trace\,(g^T(t,x) \frac{\partial^2 V}{\partial x^2}(t,x) g(t,x)$$

$$+ x(t)u(t) \int_{|z|<c} \{ V\,(\,t, x(t-) + \gamma\,(x,t)\,)$$

$$-V\big(t, x(t\,-) - \gamma_i(x,t)\big) \tfrac{\partial V}{\partial x}(x,t) \nu(dz). \tag{16}$$

where $c \in (0, \infty)$ is the maximum jump size.

Also, we define the following controlled processes [11, 14], which will appear in the proof of the exponential stability of the solution of our optimal control problem:

$$\tilde{I}_1(t) = x(t)u(t) \int_0^t \int_{|z|<c} \frac{V\,(x(s) + \xi\,(x,s,z)) - V\,(x(s))}{V\,(x(s))}$$

$$- \tfrac{\xi_i(x,s,z)}{V\,(x(s))} \, V_x\big(x(s)\big) \nu(dz) ds. \tag{17}$$

$$\tilde{I}_2(t) = x(t)u(t) \int_0^t \int_{|z|<c} (log \frac{V\,\big(x(s) + \xi\,(x,s,z)\big)}{V\,\big(x(s)\big)} + 1$$

$$-\frac{V(x(s)+\xi_i(x,s,z))}{V\left(x(s)\right)}\,\nu(dz)ds). \tag{18}$$

$$\tilde{I}(t) = x(t)u(t)\int_0^t\int_{|z|<c}\log\left(\frac{V\left(x(s)+\xi\,(x,s,z)\right)-V(x(s))}{V\left(x(s)\right)}\right.$$

$$-\frac{\xi_i(x,s,z))}{V\left(x(s)\right)}\,V_x\big(x(s)\big)\nu(dz)ds. \tag{19}$$

We note that, since $log(t) \le t-1$, we have

$$\tilde{I}_2(t) < 0,\ \forall t \ge 0. \tag{20}$$

Now, since $\tilde{I}_2 = \tilde{I}_1 - \tilde{I}$, $|\tilde{I}_2| \le |\tilde{I}_1| + |\tilde{I}|$, and $\tilde{I}(t) < \infty$, $\tilde{I}_1(t) < \infty$, we deduce the following property that guarantees the boundedness of process $\tilde{I}_2$(t) and that will be crucial in the proof of the exponential stability of the solutions of our model.

Lemma 1. Assume that the hypotheses (H3) is satisfied. Then, for all $t \ge 0$

$$\tilde{I}_2(t) < \infty. \tag{21}$$

Proof. The proof is very similar to the case presented in [11], adapted to our controlled process

$$\tilde{I}_2(t) = x(t)u(t)\int_0^t\int_{|z|<c}\left(log\frac{V\left(x(s)+\xi\,(x,s,z)\right)}{V\left(x(s)\right)}+1\right.$$

$$-\frac{V(x(s)+\xi_i(x,s,z))}{V\left(x(s)\right)}\,\nu(dz)ds). \tag{22}$$

Next, we will establish the following lemma, which is an extension of the exponential martingale inequality to Lévy case and that is a fundamental tool in the formulation of our main result over exponential stability. Its demonstration is a straightforward application of Ito's formula.

Lemma 2. (Exponential martingale inequality). Let α, β, and $T > 0$ be any positive numbers and let M(t) be a martingale, then

$$\wp\left\{sup_{0\le t\le T}\ M(t)-\frac{1}{\alpha}<M,M>(t)>\beta\right\}\le e^{-\alpha\beta}. \tag{23}$$

Proof. For the proof, see for example [11, 14].

We are now in a position to establish the next result about the boundedness of the solutions of our controlled model.

Theorem 1. Assume that the hypotheses (H1), (H2) and (H3) are satisfied. Then, there exists $K \in R$, such that

$$E\,|\,x(t)\,|^2 \le K. \tag{24}$$

Proof. We know that

$$x(t) = x_0 + \int_0^T f(x,t,u)dt + \int_0^T g(x,t,u)dW_t$$

$$+x(t)u(t) \int_0^T \int_R \xi(t,x,z)\widetilde{N}(dt,dz), \tag{25}$$

and we use the inequality:

$$|\sum_{i=1}^n x_i|^2 \le n \sum_{i=1}^n |x_i|^2, n \in N, \tag{26}$$

for $n = 4$, to obtain

$$|x(t)|^2 \le 4(|x(0)|^2 + |\int_0^T f(x,t,u)dt|^2$$
$$+ \int_0^T g(x,t,u)dW_t$$
$$+x(t)u(t) \int_0^T \int_R \xi(t,x,z)\widetilde{N}(dt,dz)|^2). \tag{27}$$

If we consider the expected value:

$$E|x(t)|^2 \le 4(E|x(0)|^2 + E|\int_0^T f(x,t,u)dt|^2$$
$$+E|\int_0^T g(x,t,u)dW_t|^2$$
$$+E|x(t)u(t) \int_0^T \int_R \xi(t,x,z)\widetilde{N}(dt,dz)|^2). \tag{28}$$

Applying the Ito-Lévy isometry we obtain:

$$E|x(t)|^2 \le 4(E|x(0)|^2 + T\int_0^T E|f(x,t,u)|^2 dt$$

$$+\int_0^T E|g(x,t,u)dt|^2$$

$$+E|x(t)u(t)|^2 \int_0^T E|\int_R \nu(dz)dt|^2). \tag{29}$$

We use (H1), (H2) and (H3) to get

$$E|x(t)|^2 \le 4(E|x(0)|^2 + T\kappa_1 \int_0^T E|x(t)|^2 dt$$
$$+\kappa_1 \int_0^T E|x(t)|^2 + \kappa_1\ \kappa_3 \int_0^T E|x(t)|^2 dt)$$
$$\le 4(E|x(0)|^2 + \kappa_1(T + 1 + \kappa_3)\int_0^T E|x(t)|^2 dt. \tag{30}$$

By using Gronwall's inequality we obtain:

$$E\,|\,x(t)\,|^{\,2} \;\le\; 4\,|\,x(0)|^{2}\;e^{\kappa_1\,(T\,+1+\kappa_3)}\,, \tag{31}$$

and taking $K \;=\; 4e^{\kappa_1\,(T\,+1+\kappa_3\,)}\;|\,x(0)\,|^{\,2}$ we arrive to

$$E\,|\,x(t)\,|^{\,2} \;\le\; K. \tag{32}$$

Now, the next theorem, which is an extension of similar result in [11] to controlled jumps, shows that under the assumptions of Theorem 1, taking $\tilde{x}(t)$ as the solution of the static problem corresponding to system (1), $x(t)$ tends exponentially to $\tilde{x}(t)$.

Theorem 2. Assume that the hypotheses (H1), (H2), and (H3) are satisfied. If there exists a Lyapunov function $V\,(x,t) \,\in\, C^{2,1}\,(\mathbb{R}^3 \,\times\, \mathbb{R}^+\,;\;\mathbb{R}^+)$, a function $\lambda(t\,:\;\mathbb{R}^+ \,\to\, \mathbb{R}^+$ that has a derivative, such that, for all t, $\lim_{t\to\infty}\lambda(t) \;=\; \infty$, and constants $\eta_2 \;>\; 0,\;\eta_3 \;>\; 0$ and $\eta_3 \;>\; 0$, such that:

a) $E\,|\,x(t) \,-\, \tilde{x}\,|^2 \;\le\; V\,(t,x), \forall t \,\in\, R^+$, where $\tilde{x}$ is the solution of the static problem corresponding to system (1).

b) $LV\,(t,x) \;\le\; \eta_2\,V\,(t,x), \forall t \,\in\, R^+$.

c) $|\,V_x^T(t,x)g(x,t,u)\,|^2 \;\le\; \eta_3\,V^2\,(t,x), \forall t \,\in\, R^+$.

d) $\tilde{I}_{\,2}(t) \le\, -\lambda(t)\eta_4\,, \forall t \,\in\, R^+$,

then, there exist a constant $r \;>\; 0$, such that

$$\lim_{t\to\infty}\frac{logE\,|\,x(t) \,-\, \tilde{x}\,|^{\,2}}{\lambda(t)} \;\le\; -r. \tag{33}$$

Proof. According to Ito's formula, the Lyapunov function $V(t,x)$ must satisfy:

$$dV(t,x) \;=\; LV(t,x)dt \;+\; V_x\,(t,x)g(t,x)\,dW$$

$$+x(t)u(t)\int_{R^3}\{V(t,x(t-) \;+\; \gamma\,(x,t)\,) \;-\; V(t,x(t-))$$

$$-\,\gamma\,(x,t)\,V_x(x,t)\;\}v(dz)\;\}$$

$$+x(t)u(t)\int_{R^3}(V(t,x(t-) \;+\; \gamma(x,t)\,)$$

$$-V(t,x(t-))\,)\widetilde{N}(dt,dz). \tag{34}$$

On the other hand, we know that:

$$dlogV(t,x) \;=\; \frac{1}{V(t,x)}(dV(t,x) - \frac{1}{2}\frac{1}{V(t,x)}dV^2(t,x)). \tag{35}$$

Therefore, we obtain

$$logV(x) = logV(x_0) + \int_0^t \frac{1}{V(x)} V_x(x) \, (f(x,s,u)ds + g(x,s,u)dW_s)$$

$$+ \frac{1}{2}\int_0^t (\frac{1}{V(x)} V_{xx} g(x,s,u) g^T(x,s,u) - \frac{1}{V^2(x)} |V_x^T g(x,s,u)|^2)ds$$

$$+ x(t)u(t)(\int_0^t \int_{|z|<c} log(V\,(x + \xi\,(t, x(t-), z)) - log(V(x))\tilde{N}(dt,dz))$$

$$+ \int_0^t \int_{|z|<c} (log(V(x + \xi\,(s, x(s-), z)) - logV(x)$$

$$- \tfrac{1}{V(x)} V_x(x)\, \xi\,(s, x(s-), z) \nu(dx) ds, \tag{36}$$

and, using the process $\tilde{I}_2\,(t)$:

$$logV(x) = logV(x_0) + \int_0^t \frac{1}{V(x)} V_x(x)(f(x,s,u)ds + g(x,s,u)dW_s\,)$$

$$+ \frac{1}{2}\int_0^t \frac{1}{V(x)} V_{xx} g(x,s,u) g^T(x,s,u)$$

$$- \frac{1}{V^2(x)} |V_x^T \; g(x,s,u)\,|^2 \;\; ds$$

$$+x(t)u(t)(\int_0^t \int_{|z|<c} log(V(s, x + \xi\,(t, x(t-)), z))$$

$$-\log\,(V(x)\tilde{N}(dt,dz))$$

$$- \frac{1}{V(x)} V_x(x)\, \xi\,(s, x(s-), z) \nu(dx) ds + \tilde{I}_2(t)$$

$$- \textstyle\int_0^t \int_{|z|<c} 1 + \frac{V(x + \xi(x,s,z))}{V(x(s))})\nu(dx)ds)\,, \tag{37}$$

and taking $M(t)$ as the martingale

$$M(t) = \frac{1}{2}\int_0^t \frac{1}{V(x(s))} V_x(x(s)) g(x,s,u) dW_s$$

$$+ x(s)u(s) \textstyle\int_0^t \int_{|z|<c} \log\,(\frac{V(x(s)) + \xi(x,s,z))}{V(x(s))} \; \tilde{N}(dt,dz)), \tag{38}$$

We can reduce (37), using $M(t)$ and the L operator:

$$logV(x(t)) \; = \; logV(x_0) \; + \int_0^t \frac{LV(x(s))}{V(x(s))} ds$$

$$-\frac{1}{2}\int_0^t \frac{1}{V^2\big(x(s)\big)} |V_x^T \;(x(s))g(x,s,u)\;|^2 \, ds$$

$$+M(t) + \tilde{I}_2(t). \tag{39}$$

On the other hand, using lemma 2, taking $\alpha \; = \; \varepsilon, \; \beta \; = \; \varepsilon n, \; for \, \varepsilon \; \in \; (0,1)$ and $n \; \in \; N, \forall n \; \geq \; t_0$:

$$\wp \{ \, sup_{t_0 \, \leq \, t \, \leq \, n} \, [M(t) - \frac{\varepsilon}{2}\int_{t_0}^t \frac{1}{V^2\big(x(s)\big)} |V_x^T g(x,s,u)\;|^2 \, ds$$

$$-x(t)u(t)\frac{1}{\varepsilon}(\int_{t_0}^t \int_{|z|<c} \left(\exp\left(\log\left(\frac{V\,(x(s) + \; \xi\;(x,s,z))}{V\big(x(s)\big)} \right)^{\varepsilon} \right) \right)$$

$$-1 - \varepsilon \log\left(\frac{V\,(x(s)+ \,\xi\,(x,s,z))}{V(x(s))} \; \nu(dz)ds \,) \right) \; > \; \varepsilon \, n \,] \leq \; exp(-\varepsilon^2 n). \tag{40}$$

We can bound $M(t)$ because of the Borel-Cantelli lemma: there exists an event $E \; \in \; \Omega$ and a random number such that for all $n \; \geq \; n_0$ the following inequality is satisfied:

$$M(t) \; \leq \frac{\varepsilon}{2}\int_{t_0}^t \frac{1}{V^2\big(x(s)\big)} |V_x^T g(x,s,u)\;|^2 \, ds$$

$$-x(t)u(t)\frac{1}{\varepsilon}\,\{\int_{t_0}^t \int_{|z|<c} \exp\left(\log\left(\frac{V\big(x(s) + \; \xi\;(x,s,z)\big)}{V\big(x(s)\big)} \right)^{\varepsilon} \right)$$

$$-1 - \; \varepsilon \; log\left(\frac{V\big(x(s)+ \,\xi(x,s,z)\big)}{V\big(x(s)\big)} \right) \nu(dz)ds \; + \; \varepsilon \, n. \tag{41}$$

Now, using (41) in (39), we obtain

$$logV(x(t)) \; \leq \; logV(x_0)) \; + \int_0^t \frac{LV\big(x(s)\big)}{V\big(x(s)\big)} \; ds + \; \tilde{I}_2\,(t)$$

$$-x(t)u(t)\frac{1}{\epsilon}\{\int_{t_0}^t \exp\left(\log\left(\frac{V\big(x(s) + \; \xi\;(x,s,z)\big)}{V\big(x(s)\big)} \right)^{\varepsilon} \right)$$

$$-1 - \varepsilon\log\,(\frac{V\big(x(s) + \; \xi(x,s,z)\big)}{V\big(x(s)\big)}))\nu(dz)ds \; + \; \varepsilon \, n$$

$$+\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\int_0^t \frac{1}{V^2\big(x(s)\big)} |V_x^T((x(s))g(x,s,u)\;|^2 \, ds. \tag{42}$$

In the central term of Eq. (42), taking $\varepsilon \to 0$ and applying, dominated convergence theorem and l'Hôpital rule, we get

$$\lim_{\epsilon\to 0} x(t)u(t)(\frac{1}{\varepsilon}\int_{t_0}^{t}\int_{|z|<c}(\exp(\log(\frac{V(x(s)+\xi(x,s,z))}{V(x(s))})^{\varepsilon}))$$

$$-1 - \varepsilon log\left(\frac{V(x(s)+\xi(x,s,z))}{V(x(s))}\nu(dz)ds\right) + \varepsilon n = 0. \quad (43)$$

So,

$$logV(x(t)) \le \log(V(x_0)) + \int_0^t \frac{LV(x(s))}{V(x(s))}ds + \tilde{I}_2(t)$$

$$+\frac{1}{2}\int_0^t \frac{1}{V^2(x(s))}|V_x^T(x(s))g(x,s,u)|^2 \, ds. \quad (44)$$

Using hypothesis b), c) and d):

$$logV(x(t)) \le logV(x_0) + \left(\eta_2 + \frac{\eta_3}{2}\right)t - \lambda(t)\eta_4. \quad (45)$$

Now, using hypothesis a)

$$\lim_{t\to\infty}\frac{logE\,|\,x(t) - \tilde{x}(t)\,|^2}{\lambda(t)} \le \lim_{t\to\infty}\frac{logV(x_0)}{\lambda(t)}$$

$$+\lim_{t\to\infty}\frac{(\eta_2 + \frac{\eta_3}{2})t}{\lambda(t)} \quad (46)$$

and by the properties of $\lambda(t)$ we have

$$\lim_{t\to\infty}\frac{logE\,|\,x(t) - \tilde{x}(t)\,|^2}{\lambda(t)} \le -\eta_4, \quad (47)$$

So, the proof is complete.

Eq. (47) says that the path of the solution $x(t)$ will converge to the path of the steady-state solution $\tilde{x}(t)$ exponentially fast.

4. Numerical Results

Numerical simulations have been carried out for the solution of the example driven by Lévy noise presented here. For the sake of simplicity of calculations, we assume that the driving Poisson random measures are generated by stationary Poisson point processes. So, we just an approximation of Lévy processes. According to the parameters on (9) and (10), the values chosen are $\beta = 0.1$, $\delta = 0.3$, $\gamma = 0.2$, $\varepsilon = 0.4$, $\omega = 0.5$, $\kappa = 0.6$, A1 = 0.3, A2 = 0.2, A3 = 0.07. In Fig. 1 we present limit trajectory of the states $x_1(t)$, $x_2(t)$and $x_3(t)$, using the Euler-Maruyama scheme.

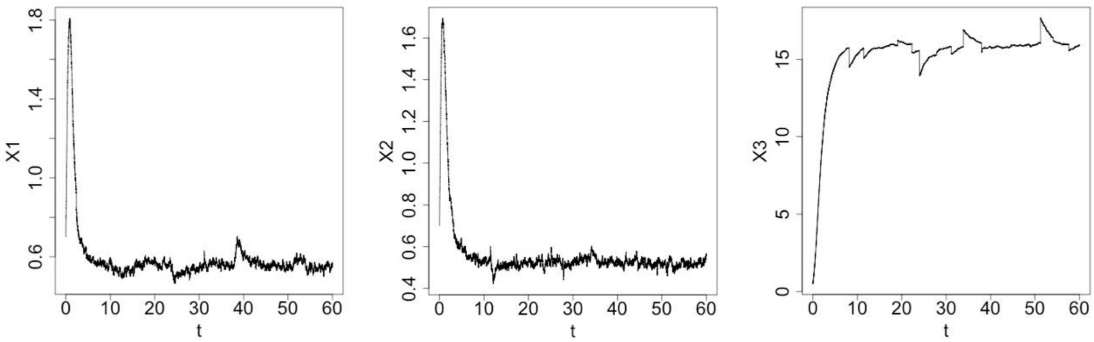

Fig. 1. Stochastic states limit trajectory $x_1(t)$, $x_2(t)$ and $x_3(t)$ with white noise and Lévy jumps.

5. Final Considerations and Conclusions

In this paper, we have analyzed one kind of Lotka-Volterra systems which represent one of the most important models to describe population dynamics in competition. This is a controlled stochastic Lotka-Volterra model for one-predator-two-preys. That system contains control functions, white noise and Lévy jumps.

We have studied the boundedness and the exponential stability of the solutions of the system, around the static solution. We have shown that the solutions of this controlled stochastic Lotka-Volterra model with Lévy jumps are bounded and exponentially stable. In our context, it is important to develop some techniques to control random fluctuations and sudden end unpredictable changes in the environment, like Lyapunov approach or geometric techniques of Control Theory, as part of an optimal and general strategy for the preservation of species and the harvest of any renewable resource as some animals or plants. In other contexts, where the species may be viruses or even subatomic particles, this approach is equally useful.

Although we do not present a sensitivity analysis of the parameters associated with Lévy jumps and white noise, from our numerical simulations, we find that, if the noise perturbation is small (α_1, α_1, $\alpha_1 <$ 0.20) and ξ = 0.013284, for Lévy jumps, then the exponential stability property is preserved in the stochastic model, despite the two kinds of disturbance considered.

Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper

Author Contributions

All authors contributed equally and significantly in writing this paper. Cutberto Romero-Meléndez conceived the research concept and framework, conducted the study design, and reviewed and finalized the manuscript for the final version. David Castillo-Fernández contributed to the mathematical analysis of the model, including stability proofs and the design of numerical simulations. Leopoldo González-Santos assisted with the literature review, participated in discussions, and provided professional insights, especially regarding the application of the model in a neurobiological context. All authors were actively involved in drafting the manuscript, provided significant academic input, and approved the final version of the manuscript for submission.

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