Generating Functions for k-Hypergeometric Functions

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Abstract: In mathematics, generating functions are important way to transform formal power series into functions and to analyze asymptotic properties of sequences. On the other hand, hypergeometric function is a special function represented by the hypergeometric series. Gauss, Confluent, Appell, Lauricella, and Horn functions are as an example of hypergeometric functions. In this article, we will introduce k-hypergeometric functions which are extensions of the Gauss hypergeometric functions including k- Pochhammer symbol. We first give an identity for k-Pochhammer symbol and certain linear generating functions for k-hypergeometric functions. Then we derive a family of multilinear and multilateral generating functions for these functions. In the main theorems, specially, bilateral generating function relations are obtained by applying extended multivariable hypergeometric functions and Cesáro Polynomials. Additionally we present bilinear generating functions for k-hypergeometric functions.

Key words: Hypergeometric functions, generating functions, multilinear and multilateral generating functions, k-hypergeometric functions, *k*-Pochhammer symbol.

1. Introduction

The hypergeometric functions are important for obtaining various properties, such as, integral representation, generating functions, solution of Gauss differential equations.

We aim at deriving some generating functions for a family of the *k*-hypergeometric functions defined by (see [1]):

$$ {}_2F_{1,k} = F_k(\alpha,\beta;\gamma;z) := \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}\, n!} z^n \tag{1}$$
$$(\alpha,\beta,\gamma \in C; \quad \gamma \neq 0,-1,-2,\dots; \ |z|<1 \ \ and \ \ k>0)$$

in terms of the following Pochhammer *k*-symbol $(\lambda)_{\nu,k}$ $(\lambda,\nu \in C)$ defined by [2]:

$$\begin{aligned}(\lambda)_{\nu,k} &:= \frac{\Gamma_k(\lambda+\nu k)}{\Gamma_k(\lambda)} \\ &= \begin{cases} 1 & ;\nu=0, \ \ \lambda \in C-\{0\} \\ \lambda(\lambda+k)(\lambda+2k)\dots(\lambda+(n-1)k) & ;\nu=n,\ n\in N,\ k>0 \ \lambda \in C-kZ^- \end{cases}\end{aligned} \tag{2}$$

where the k-gamma function $\Gamma_k(z)$ was introduced in [2] as follows:

$$\Gamma_k(z) := \int_0^{\infty} t^{z-1} e^{\frac{-t^k}{k}} dt \qquad \mathrm{Re}(z) > 0. \tag{3}$$

Here, and in what follows, we respectively denote by R and C the sets of real and complex numbers and $N_0 := N \cup \{0\}$, $(N = 1, 2, 3, \ldots)$ and $Z := Z^- \cup \{0\}$, $(Z^- = -1, -2, -3, \ldots)$.

Clearly, in the special case $k = 1$ equations (1), (2), and (3) are reduces to the usual hypergeometric function, Pochhammer symbol, and gamma function given by respectively [3]:

$$F(\alpha, \beta; \gamma; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \, n!} z^n; \qquad (|z| < 1),$$

$$(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & ;\nu = 0, \quad \lambda \in C?\{\ \} \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+(n-1)) & ;\nu = n, \ n \in N, \ \lambda \in C \end{cases},$$

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \qquad \mathrm{Re}(z) > 0.$$

The main aim of this paper is to derive various classes of multilinear and multilateral generating functions for k-hypergeometric functions given by (1). We also give special cases of generating functions presented in this article.

2. Generating Functions

In this section, we derive a property of k-Pochhammer symbol and obtain generating function relations for k-hypergeometric functions.

Lemma 2.1. We have following identity for k-Pochhammer symbol given by (2):

$$\frac{(-n)_{r,k}}{(-1)^r} = \frac{k^r \left(\frac{n}{k}\right)!}{\left(\frac{n}{k} - r\right)!}.$$

Proof. If we use k-Pochhammer symbol definition given by (2), then

$$\begin{aligned} (-n)_{r,k} &= (-n)(-n+k)(-n+2k)\ldots(-n+k(r-1)) \\ &= \frac{(-1)^r k^r \left(\frac{n}{k}\right)\left(\frac{n}{k}-1\right)\ldots\left(\frac{n}{k}-(r-1)\right)\left(\frac{n}{k}-r\right)\ldots 321}{\left(\frac{n}{k}-r\right)\ldots 321} \\ &= \frac{(-1)^r k^r \left(\frac{n}{k}\right)!}{\left(\frac{n}{k}-r\right)!} \end{aligned}$$

which completes the proof.

Theorem 2.1. *We have the following generating function for the k-hypergeometric functions given by (1):*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_{n,k}}{n!}F_k\left(-nk,\beta;\gamma;x\right)t^n=\left(1-kt\right)^{\frac{-\lambda}{k}}F_k\left(\lambda,\beta;\gamma;\frac{-kxt}{1-kt}\right) \quad (4)$$

where $\lambda\in C$ *and* $|t|<1.$

Proof. Let T denote the first member of assertion (4). Using (1), we have

$$T=\sum_{n=0}^{\infty}\frac{(\lambda)_{n,k}}{n!}\sum_{r=0}^{r<nk}\frac{(-nk)_{r,k}(\beta)_{r,k}}{(\gamma)_{r,k}r!}x^rt^n=\sum_{n=0}^{\infty}\sum_{r=0}^{r<nk}\frac{(\lambda)_{n,k}(-nk)_{r,k}(\beta)_{r,k}}{(\gamma)_{r,k}n!r!}x^rt^n$$

and using Lemma 2.1, we get,

$$T=\sum_{n=0}^{\infty}\sum_{r=0}^{r<nk}\frac{(\lambda)_{n,k}(\beta)_{r,k}(-1)^rk^rn!}{(\gamma)_{r,k}n!r!(n-r)!}x^rt^n.$$

Replacing n by $n+r$ we may write that

$$T=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\lambda)_{n+r,k}(\beta)_{r,k}(-1)^rk^r}{(\gamma)_{r,k}n!r!}x^rt^{n+r}.$$

Using the following properties (see [4], [5])

$$(\lambda)_{n+r,k}=(\lambda)_{r,k}(\lambda+rk)_{n,k},\quad \sum_{n=0}^{\infty}\frac{(\lambda)_{n,k}}{n!}t^n=\left(1-kt\right)^{-\frac{\lambda}{k}}$$

we get

$$T=\left(1-kt\right)^{-\frac{\lambda}{k}}\sum_{r=0}^{\infty}\frac{(\lambda)_{r,k}(\beta)_{r,k}}{(\gamma)_{r,k}r!}\left(\frac{-kxt}{1-kt}\right)^r=\left(1-kt\right)^{-\frac{\lambda}{k}}F_k\left(\lambda,\beta;\gamma;\frac{-kxt}{1-kt}\right),$$

which completes the proof.

Theorem 2.2. *We have the following generating function for the k-hypergeometric functions given by (1):*

$$\sum_{n=0}^{\infty}F_k(-nk,\beta;\gamma;x)\frac{t^n}{n!}=e^t\,\Phi_k(\beta;\gamma;-kxt) \quad (5)$$

where $|t|<1$ *and* Φ_k *is k-confluent hypergeometric function.*

Proof. Let T denote the first member of assertion (5). Using (1), we have

$$T=\sum_{n=0}^{\infty}\sum_{r=0}^{r<nk}\frac{(-nk)_{r,k}(\beta)_{r,k}}{(\gamma)_{r,k}r!}\frac{x^rt^n}{n!}$$

and using Lemma 2.1, we get,

$$T = \sum_{n=0}^{\infty} \sum_{r=0}^{r<nk} \frac{(\beta)_{r,k} (-1)^r k^r n!}{(\gamma)_{r,k}\, n! r!(n-r)!} x^r t^n.$$

Replacing n by $n+r$ we may write that

$$T = \sum_{r=0}^{\infty} \frac{(\beta)_{r,k}}{(\gamma)_{r,k}\, r!} (-kxt)^r \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t\, \Phi_k\left(\beta;\gamma;-kxt\right),$$

which completes the proof.

3. Multilinear and Multilateral Generating Functions

In this section, we derive several families of bilinear and bilateral generating functions for the k-hypergeometric functions by using the similar method considered in [6]-[9].

Theorem 3.1. Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,\ldots,y_s)$ of s complex variables $y_1,\ldots,y_s\ (s\in N)$ and of complex order μ, let

$$\Lambda_{\mu,\psi}(y_1,\ldots,y_s;\xi) := \sum_{l=0}^{\infty} a_l\, \Omega_{\mu+\psi l}(y_1,\ldots,y_s)\xi^l$$

where $a_l \neq 0,\ \mu,\psi \in C$ *and*

$$\Theta_{n,p}^{\mu,\psi}(x,y_1,\ldots,y_s;\zeta) := \sum_{l=0}^{[n/p]} a_l\, (\lambda)_{n-pl,k}\, F_k(-(n-pl)k,\beta;\gamma;x)\,\Omega_{\mu+\psi l}(y_1,\ldots,y_s)\frac{\zeta^l}{(n-pl)!}$$

Then, for $p\in N$ we have

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(x,y_1,\ldots,y_s;\frac{\eta}{t^p}\right)t^n = (1-kt)^{-\frac{\lambda}{k}} F_k\left(\lambda,\beta;\gamma;\frac{-kxt}{1-kt}\right)\Lambda_{\mu,\psi}(y_1,\ldots,y_s;\eta) \tag{6}$$

provided that each member of (6) exists.

Proof. For convenience, let S denote the first member of the assertion (6). Then,

$$S = \sum_{n=0}^{\infty}\sum_{l=0}^{[n/p]} a_l\, (\lambda)_{n-pl,k}\, F_k(-(n-pl)k,\beta;\gamma;x)\,\Omega_{\mu+\psi l}(y_1,\ldots,y_s)\frac{\eta^l\, t^{n-pl}}{(n-pl)!}$$

Replacing n by $n+pl$ we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} a_l\, (\lambda)_{n,k}\, F_k(-nk,\beta;\gamma;x)\,\Omega_{\mu+\psi l}(y_1,\ldots,y_s)\frac{\eta^l t^n}{n!} \\ &= \sum_{n=0}^{\infty} (\lambda)_{n,k}\, F_k(-nk,\beta;\gamma;x)\frac{t^n}{n!}\sum_{l=0}^{\infty} a_l \Omega_{\mu+\psi l}(y_1,\ldots,y_s)\eta^l \\ &= (1-kt)^{-\frac{\lambda}{k}} F_k\left(\lambda,\beta;\gamma;\frac{-kxt}{1-kt}\right)\Lambda_{\mu,\psi}\left(y_1,\ldots,y_s;\eta\right), \end{aligned}$$

which completes the proof. By using a similar idea, we also get the next result immediately.

Theorem 3.2. Corresponding to an identically non-vanishing function $\Omega_{\mu}(y_1,\ldots,y_s)$ of s complex variables $y_1,\ldots,y_s\ (s\in N)$ and of complex order μ, let

$$\Lambda_{\mu,\psi}(y_1,\ldots,y_s;\xi):=\sum_{l=0}^{\infty}a_l\,\Omega_{\mu+\psi l}(y_1,\ldots,y_s)\xi^l$$

where $a_l\neq 0,\ \mu,\psi\in C$ and

$$\Theta_{n,p}^{\mu,\psi}(x,y_1,\ldots,y_s;\zeta):=\sum_{l=0}^{[n/p]}a_lF_k(-(n-pl)k,\beta;\gamma;x)\Omega_{\mu+\psi l}(y_1,\ldots,y_s)\frac{\zeta^l}{(n-pl)!}$$

Then, for $p\in N$ we have

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x,y_1,\ldots,y_s;\frac{\eta}{t^p}\right)t^n=e^t\,\Phi_k\left(\beta;\gamma;-kxt\right)\Lambda_{\mu,\psi}(y_1,\ldots,y_s;\eta)\tag{7}$$

provided that each member of (7) exists.

4. Special Cases

As an application of the above theorems, when the multivariable function $\Omega_{\mu}(y_1,\ldots,y_s)$, $l\in N_0$, $s\in N$ is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$s=r,\quad \Omega_{\mu+\psi l}(y_1,\ldots,y_r)={}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0;p,q}\right)}(\lambda+\mu+\psi l,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;y_1,\ldots,y_r)$$

in Theorem 3.1, where the extended multivariable hypergeometric functions ${}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0;p,q}\right)}$ generated by (see [10]):

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0;p,q}\right)}\left(\lambda+n,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;x_1,\ldots,x_r\right)t^n$$
$$=(1-t)^{-\lambda}\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0;p,q}\right)}\left(\lambda,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;\frac{x_1}{(1-t)^{\rho}},\ldots,\frac{x_k}{(1-t)^{\rho}},\frac{x_{k+1}}{(1-t)},\ldots,\frac{x_r}{(1-t)}\right).\tag{8}$$

We are thus led to the following result which provides a class of bilateral generating functions for k-hypergeometric functions and the extended multivariable hypergeometric functions.

Corollary 4.1. *If*

$$\Lambda_{\mu,\psi}(y_1,\ldots,y_s;\xi):=\sum_{l=0}^{\infty}a_l\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0;p,q}\right)}(\lambda+\mu+\psi l,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;y_1,\ldots,y_r)\xi^l$$
$$(a_l\neq 0,\ \mu,\psi\in C)$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{l=0}^{[n/p]} a_l\,(\lambda)_{n-pl,k}\,F_k\left(\begin{matrix}-(n-pl)k,\beta\\ \gamma\end{matrix};x\right)\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0;p,q}\right)}\left(\begin{matrix}\lambda+\mu+\psi l,\beta_{k+1},\dots,\beta_r\\ \gamma_1,\dots,\gamma_r\end{matrix};y_1,\dots,y_r\right)\frac{\eta^l\,t^{n-pl}}{(n-pl)!}$$

$$=(1-kt)^{-\frac{\lambda}{k}}\,F_k\left(\lambda,\beta;\gamma;\frac{-kxt}{1-kt}\right)\Lambda_{\mu,\psi}(y_1,\dots,y_r;\eta) \tag{9}$$

provided that each member of (9) exists.

Remark 4.1. Using the generating relation (8) for the extended multivariable hypergeometric functions and getting $a_l=\dfrac{(\lambda)_l}{l!}$, $\mu=0,\ \psi=1$ in Corollary 4.1, we find that

$$\sum_{n=0}^{\infty}\sum_{l=0}^{[n/p]}\frac{(\lambda)_l}{l!}(\lambda)_{n-pl,k}\,F_k\left(\begin{matrix}-(n-pl)k,\beta\\ \gamma\end{matrix};x\right)\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0;p,q}\right)}\left(\begin{matrix}\lambda+l,\beta_{k+1},\dots,\beta_r\\ \gamma_1,\dots,\gamma_r\end{matrix};y_1,\dots,y_r\right)\frac{\eta^l\,t^{n-pl}}{(n-pl)!}$$

$$=(1-kt)^{-\frac{\lambda}{k}}\,F_k\left(\lambda,\beta;\gamma;\frac{-kxt}{1-kt}\right)(1-t)^{-\lambda}$$

$$\times\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0;p,q}\right)}\left(\lambda,\beta_{k+1},\dots,\beta_r;\gamma_1,\dots,\gamma_r;\frac{x_1}{(1-t)^{\rho}},\dots,\frac{x_k}{(1-t)^{\rho}},\frac{x_{k+1}}{(1-t)},\dots,\frac{x_r}{(1-t)}\right).$$

On the other hand, we set $\Omega_{\mu+\psi l}(y)=F_k(-(\mu+\psi l)k,\beta;\gamma;y)$ in Theorem 3.2, we have the bilinear generating function relations for the *k*-hypergeometric functions.

Corollary 4.2. *If*

$$\Lambda_{\mu,\psi}(y;\xi):=\sum_{l=0}^{\infty}a_l\,F_k(-(\mu+\psi l)k,\beta;\gamma;y)\xi^l$$
$$(a_l\neq 0,\ \mu,\psi\in C)$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{l=0}^{[n/p]}a_l F_k\left(\begin{matrix}-(n-pl)k,\beta\\ \gamma\end{matrix};x\right)F_k\left(\begin{matrix}-(\mu+\psi l)k,\beta\\ \gamma\end{matrix};y\right)\frac{\eta^l\,t^{n-pl}}{(n-pl)!}$$
$$=e^t\,\Phi_k(\beta;\gamma;-kxt)\Lambda_{\mu,\psi}(y;\eta) \tag{10}$$

provided that each member of (10) exists.

Remark 4.2. Using the generating relation (5) for the k-hypergeometric functions and getting $a_l=\dfrac{1}{l!}$, $\mu=0,\ \psi=1$ in Corollary 4.2, we find that

$$\sum_{n=0}^{\infty}\sum_{l=0}^{[n/p]}F_k\left(\begin{matrix}-(n-pl)k,\beta\\ \gamma\end{matrix};x\right)F_k\left(\begin{matrix}-lk,\beta\\ \gamma\end{matrix};y\right)\frac{\eta^l\,t^{n-pl}}{(n-pl)!\,l!}=e^{2t}\,\Phi_k(\beta;\gamma;-kxt)\Phi_k(\beta;\gamma;-ky\eta).$$

In the same way, we set, $\Omega_{\mu+\psi l}(y) = F_k(-(\mu+\psi l)k, \beta; \gamma; y)$ in Theorem 3.1 we have the bilinear generating function relation for the k-hypergeometric functions. Then again, we set, $\Omega_{\mu+\psi l}(y) = g^{(s)}_{\mu+\psi l}(y)$ in Theorem 3.2, where the Cesáro polynomials generated by [8], [9]:

$$\sum_{n=0}^{\infty} g_n^{(s)}(x)t^n = (1-t)^{-s-1}(1-xt)^{-1}. \tag{11}$$

We are thus led to the following result which provides a class of bilateral generating functions for k-hypergeometric functions and the Cesáro polynomials.

Corollary 4.3. If

$$\Lambda_{\mu,\psi}(y;\xi) := \sum_{l=0}^{\infty} a_l\, g^{(s)}_{\mu+\psi l}(y)\xi^l$$
$$(a_l \neq 0,\ \mu,\psi \in C)$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{l=0}^{[n/p]} a_l F_k\left(\begin{matrix}-(n-pl)k,\beta \\ \gamma\end{matrix};x\right) g^{(s)}_{\mu+\psi l}(y)\frac{\eta^l\, t^{n-pl}}{(n-pl)!} = e^t\, \Phi_k(\beta;\gamma;-kxt)\Lambda_{\mu,\psi}(y;\eta) \tag{12}$$

provided that each member of (12) exists.

Remark 4.3. Using the generating relation (11) for the Cesáro polynomials and getting $a_l = 1,\ \mu = 0,\ \psi = 1$ in Corollary 4.3, we find that

$$\sum_{n=0}^{\infty}\sum_{l=0}^{[n/p]} F_k\left(\begin{matrix}-(n-pl)k,\beta \\ \gamma\end{matrix};x\right) g^{(s)}_{l}(y)\frac{\eta^l\, t^{n-pl}}{(n-pl)!} = e^t\, \Phi_k(\beta;\gamma;-kxt)(1-\eta)^{-s-1}(1-y\eta)^{-1}.$$

Furthermore, for every suitable choice of the coefficients $a_i\,(i \in N_0)$ if the multivariable function $\Omega_{\mu+\psi l}(y_1,\ldots,y_s)$ $s \in N$ is expressed as an appropriate product of several simpler functions, the assertions of Theorems 3.1 and 3.2 can be applied in order to derive various families of multilinear and multilateral generating functions for the k-hypergeometric functions.

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