Multiplicative (Generalized)-(α, β)-Derivations in Prime and Semiprime Rings

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Abstract: Let R be an associative ring and α, β be the automorphisms on R. A map $F: R \to R$ (not necessarily additive) is said to be multiplicative (generalized)-(α, β)-derivation if it satisfies $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all x, y ∈ R, where d is any map on R. Suppose that G and F are multiplicative (generalized)-(α, β)-derivations associated with the maps g and d on R respectively. The main aim in this article is to study the following situations: (i) $G(xy) - F(x)F(y) = 0$; (ii) $F(xy) + \alpha(xy) = 0$; $(iii) F(x)F(y) + \alpha(xy) = 0$ for all x, y in some appropriate subsets of prime or semiprime ring R.

Key words: Left ideal, lie ideal, multiplicative (generalized)-(α,β)-derivation, prime ring.

1. Introduction

Throughout the present paper, R will denote an associative ring with center $Z(R)$. For given $x, y \in R$, the symbols $[x,y]$ and xoy denote the commutator $xy - yx$ and anti-commutator $xy + yx$ respectively. For any $x \in R$, if $2x = 0$ implies that $x = 0$, then R is said to be 2-torsion free ring. Recall that a ring R is said to be prime, whenever $aRb = (0)$ with $a, b \in R$ implies either $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. An additive subgroup U of R is called Lie ideal of ring R, if $[u, r] \in U$ for all $u \in U, r \in R$ and is called square closed if square of every element of U is in U. Moreover if U is square closed Lie ideal of R, then $2xy \in U$ for all $x, y \in U$.

An additive mapping $d: R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$ is called derivation. Bresar [1] first introduced the concept of generalized derivation. An additive map $F: R \to R$ associated with a derivation d on R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, is said to be generalized derivation. Let $a, b \in R$, an additive mapping $F: R \to R$ defined as $F(x) = ax + xb$ for all $x \in R$ is an example of generalized derivation.

A mapping $F: R \to R$ (not necessarily additive) is called a multiplicative (generalized)- derivation if it satisfies $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is any map on R. Obviously, every generalized derivation is a multiplicative (generalized)-derivation on R. A map $F: R \to R$ (not necessarily additive) is said to be multiplicative (generalized)-(α, β)-derivation if it satisfies $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$, where d is any map on R.

Recently, Dhara and Ali [2]; Ali *et al.* [3]; Dhara *et al.* [4]; Khan [5] investigated the multiplicative (generalized)-derivations satisfying certain identities on some suitable subsets in the prime and semiprime rings. In the present paper, our aim is to investigate some algebraic identities involving multiplicative (generalized)-(α, β)-derivations on some suitable subsets in prime and semiprime rings.

2. Preliminary Results

The following lemma will be used in the main results.

Lemma 2.1 If U not contained in R, is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aUb = (0)$, then $a=0$ or $b = 0$.

Proof: It is proved in [[6], Lemma 4].

3. Main Results

Theorem 3.1 Let R be a semiprime ring, L be a nonzero left ideal of R and α be any epimorphism on R. Suppose that F and G are two multiplicative (generalized)-(α, α)-derivations on R associated with the maps d and g on R. If $G(xy) - F(x)F(y) = 0$ for all $x, y \in L$, then $\alpha(L)d(L) = (0)$, $\alpha(L)g(L) = (0)$, $\alpha(L)[F(x), \alpha(x)] = (0)$ and $\alpha(L)[G(x), \alpha(x)] = (0)$ for all $x \in L$.

Proof: We have the identity

$$G(xy) - F(x)F(y) = 0 \tag{1}$$

for all $x, y \in L$. Replacing y by yz in (1), we obtain

$$G(xy)\alpha(z) + \alpha(xy)g(z) - F(x)F(y)\alpha(z) - F(x)\alpha(y)d(z) = 0 \tag{2}$$

for all $x, y, z \in L$. Using (1) in (2), we get

$$\alpha(x)\alpha(y)g(z) - F(x)\alpha(y)d(z) = 0 \tag{3}$$

for all $x, y, z \in L$. Replacing x by xw in (3), we have

$$\alpha(x)\alpha(w)\alpha(y)d(z) - F(x)\alpha(w)\alpha(y)d(z) - \alpha(x)d(w)\alpha(y)d(z) = 0 \tag{4}$$

for all $x, y, z, w \in L$. Again, substituting y by wy in (3), we obtain

$$\alpha(x)\alpha(w)\alpha(y)d(z) - F(x)\alpha(w)\alpha(y)d(z) = 0 \tag{5}$$

for all $x, y, z, w \in L$. Subtracting (4) from (5), we find

$$\alpha(x)d(w)\alpha(y)d(z) = 0 \tag{6}$$

for all $x, y, z, w \in L$. Since L is a left ideal and α is an epimorphism, so from (6), we get $\alpha(x)d(w)R\alpha(y)d(z) = (0)$. In particular, $\alpha(x)d(y)R\alpha(x)d(y) = (0)$ for all $x, y \in L$. By semiprimeness of R, we have $\alpha(L)d(L) = (0)$. Using $\alpha(L)d(L) = (0)$ in (3), we obtain

$$\alpha(x)\alpha(y)g(z) = 0 \tag{7}$$

for all $x, y, z \in L$. Again, L is a left ideal and α is an epimorphism, so from (7), we find $\alpha(x)R\alpha(y)g(z) = (0)$, that is $\alpha(x)r\alpha(y)g(z) = 0$, where $r \in R$. Replacing r by $g(w)r$ in the last expression, we get $\alpha(x)g(w)r\alpha(y)g(z) = 0$. In particular, $\alpha(x)g(y)R\alpha(x)g(y) = (0)$ for all x, y $\in$ L. By semiprimeness of R, we have $\alpha(L)g(L) = (0)$. Thus we obtain $F(xy) = F(x)\alpha(y)$ and $G(xy) = G(x)\alpha(y)$. Now, from equation (1),

$$G(xyz) = F(xy)F(z) = F(x)F(yz) \tag{8}$$

for all $x, y, z \in L$. From (8), we write $F(x)\alpha(y)F(z) = F(x)F(y)\alpha(z)$. In particular, $F(x)[F(y), \alpha(y)] = 0$. Substituting x by xw in the last expression, we have $F(x)\alpha(w)[F(y), \alpha(y)] = 0$. Since L is a left ideal or R, it follows that $[F(x), \alpha(x)]\alpha(w)[F(y), \alpha(y)] = 0$ which yields $[F(x), \alpha(x)]R\alpha(w)[F(y), \alpha(y)] = (0)$, so $\alpha(w)[F(x), \alpha(x)]R\alpha(w)[F(y), \alpha(y)] = (0)$.

In particular, $\alpha(L)[F(y), \alpha(y)]R\alpha(L)[F(y), \alpha(y)] = (0)$ holds for all $y \in L$. Semiprimeness of R forces to write $\alpha(L)[F(x), \alpha(x)] = (0)$ for all $x \in L$. Using equation (1) and $\alpha(L)[F(x), \alpha(x)] = (0)$, we find

$$\begin{aligned}\alpha(z)G(x)\alpha(xy) &= \alpha(z)G(x^2 y) = \alpha(z)F(x^2)F(y) = \alpha(z)F(x)\alpha(x)F(y) \\ &= \alpha(z)\alpha(x)F(x)F(y) = \alpha(z)\alpha(x)G(xy) = \alpha(z)\alpha(x)G(x)\alpha(y)\end{aligned} \tag{9}$$

for all x, y, z $\in$ L. Equation (9) gives $\alpha(z)[G(x), \alpha(x)]\alpha(y) = 0$. Since L is a left ideal, it follows that $\alpha(z)[G(x), \alpha(x)]R\alpha(y) = (0)$. Thus $\alpha(L)[G(x), \alpha(x)]R\alpha(L)[G(x), \alpha(x)] = (0)$. By the semiprimeness of R, we have $\alpha(L)[G(x), \alpha(x)] = (0)$ for all $x \in L$. Thereby the proof of the theorem is completed.

Corollary 3.2 ([4], Theorem 3.1). Let R be a semiprime ring, L be a nonzero left ideal of R and α be any epimorphism on R. Suppose that F is a multiplicative (generalized)-(α, α)-derivation on R associated with the map d on R. If $F(xy) - F(x)F(y) = 0$ for all $x, y \in L$, then $\alpha(L)d(L) = (0)$ and $\alpha(L)[F(x), \alpha(x)] = (0)$ for all $x \in L$.

Proof: Replacing G by F and g by d in theorem 3.1, we get the required result.

Theorem 3.3 Let R be a 2-torsion free prime ring, U be a nonzero square closed Lie ideal of R and α, β be the automorphisms on R. Suppose that F is a multiplicative (generalized)-(α, β)-derivation on R associated with the map d on R. If $F(xy) + \alpha(xy) = 0$ for all $x, y \in U$, then $F(x) = -\alpha(x)$ for all $x \in U$ and $d(U) = (0)$.

Proof: We have the identity

$$F(xy) + \alpha(xy) = 0 \tag{10}$$

for all $x, y \in U$. Replacing y by $2yz$ in (10) and using 2-torsion freeness of R, we get

$$F(xy)\alpha(z) + \beta(xy)d(z) + \alpha(xy)\alpha(z) = 0 \tag{11}$$

for all $x, y, z \in U$. Using (10) in (11), we have

$$\beta(x)\beta(y)d(z) = 0 \tag{12}$$

for all $x, y, z \in U$. Replacing x by $[r, x]$ in (12), where $r \in R$, we obtain $\beta(x)\beta(r)\beta(y)d(z) = 0$. Since U is nonzero Lie ideal, so by primeness of R, we find $\beta(y)d(z) = 0$. Replacing y by $[y, r]$ in the last expression, where $r \in R$, we find $\beta(y)\beta(r)d(z) = 0$. Thus, we get $d(U) = (0)$. Now, from (10), $F(x)\alpha(y) + \alpha(xy) = 0$, that is $\{F(x) + \alpha(x)\}\alpha(y) = 0$. Substituting y by $[r, y]$ in last expression, where $r \in R$, we have $\{F(x) + \alpha(x)\}\alpha(r)\alpha(y) = 0$. By primeness of R, we conclude that $F(x) = -\alpha(x)$ for all $x \in U$. Hence, the proof of the theorem is completed.

Corollary 3.4 ([3], Theorem 3.4). Let R be a 2-torsion free prime ring, U be a nonzero square closed Lie ideal of R and α, β be the automorphisms on R. Suppose that F is a multiplicative (generalized)-derivation on R associated with the map d on R. If $F(xy) + xy = 0$ for all $x, y \in U$, then $F(x) = -x$ for all $x \in U$ and $d(U) = (0)$.

Proof: Replacing α and β by *1* in theorem 3.3, where *1* is the identity map on R, we get the required result.

Theorem 3.5 Let R be a 2-torsion free prime ring, U be a nonzero square closed Lie ideal of R and α, β be the automorphisms on R. Suppose that F is a multiplicative (generalized)-(α, β)-derivation on R associated with the map d on R. If $F(x)F(y) + \alpha(xy) = 0$ for all $x, y \in U$, then $[F(x), \alpha(x)] = 0$ for all $x \in U$ and $d(U) = (0)$.

Proof: We have the identity

$$F(x)F(y) + \alpha(xy) = 0 \tag{13}$$

for all $x, y \in U$. Replacing y by $2yz$ in (13) and using 2-torsion freeness of R, we get

$$F(x)F(y)\alpha(z) + F(x)\beta(y)d(z) + \alpha(xy)\alpha(z) = 0 \tag{14}$$

for all $x, y, z \in U$. Using (13) in (14), we have

$$F(x)\beta(y)d(z) = 0 \tag{15}$$

for all $x, y, z \in U$. Left multiply by $F(w)$ to (15), we obtain $F(w)F(x)\beta(y)d(z) = 0$ for all $x, y, z, w \in U$. Equation (13) yields that $\alpha(w)\alpha(x)\beta(y)d(z) = 0$. Replacing w by $[w, r]$, where $r \in R$, we find $\alpha(w)\alpha(r)\alpha(x)\beta(y)d(z) = 0$. Since U is nonzero Lie ideal of R, so primeness of R forces to write $\alpha(x)\beta(y)d(z) = 0$. Repeating the same technique twice, we get $d(U) = (0)$. Now, substituting y by y^2 in (13) and using $d(U) = (0)$, we have

$$F(x)F(y)\alpha(y) + \alpha(xy^2) = 0 \tag{16}$$

for all $x, y \in U$. Again, replacing x by $2xy$ in (13) and using 2-torsion freeness of R, we obtain

$$F(x)\alpha(y)F(y) + \alpha(xy^2) = 0 \tag{17}$$

for all $x, y \in U$. Subtracting (16) from (17), we find $F(x)[F(y), \alpha(y)] = 0$. Replacing x by 2xw in the last expression and using 2-torsion freeness of R, we get $F(x)\alpha(w)[F(y), \alpha(y)] = 0$ for all $x, y, w \in U$.

It follows that $[F(x), \alpha(x)]\alpha(w)[F(y), \alpha(y)] = 0$ for all $x, y, w \in U$. In particular, $[F(y), \alpha(y)]\alpha(w)[F(y), \alpha(y)] = 0$. If U is not contained in R, then lemma 2.1 forces to write that $[F(x), \alpha(x)] = 0$ for all $x \in U$ and if $U \subseteq R$, then $[F(x), \alpha(x)] = 0$ for all $x \in U$ holds trivially. This completes the proof.

Corollary 3.6 ([3], Theorem 3.6). Let R be a 2-torsion free prime ring, U be a nonzero square closed Lie ideal of R and α, β be the automorphisms on R. Suppose that F is a multiplicative (generalized)-derivation on R associated with the map d on R. If $F(x)F(y) + xy = 0$ for all $x, y \in U$, then $[F(x), x] = 0$ for all $x \in U$ and $d(U) = (0)$.

Proof: Replacing α and β by 1 in theorem 3.5, where 1 is the identity map on R, we get the required result.

Now, we conclude this paper with an example which shows that the semiprimeness of the ring in our results is essential.

Example Consider The ring $R = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in X\right\}$, where X is the set of integers. Let $L = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in X\right\}$ be a left ideal of R. Let $\alpha : R \to R$ defined by $\alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$ be the automorphism. Define mapping d and F on R such that $d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b/2 \\ 0 & 0 \end{pmatrix}$ and $F \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b/2 \\ 0 & 0 \end{pmatrix}$ respectively. Again define mapping g and G on R such that $g \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b/2 \\ 0 & 0 \end{pmatrix}$ and $G \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b/2 \\ 0 & c \end{pmatrix}$ respectively. We notice that F and G both are multiplicative (generalized)-(α, α)-derivations on R associated with the maps d and g on R respectively. It is straightforward to verify that $G(xy) - F(x)F(y) = 0$ for all $x, y \in L$. We see that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0)$, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nonzero element of R. It implies that R is not semiprime ring. In this example, we note that $\alpha(L)d(L) \neq (0)$, $\alpha(L)g(L) \neq (0)$, $\alpha(L)[F(x), \alpha(x)] \neq (0)$ and $\alpha(L)[G(x), \alpha(x)] \neq (0)$ for some $x \in L$.

4. Conclusion

We know that derivations on rings are closely connected with the behavior of rings. Many authors investigated the commutativity in prime and semiprime rings admitting derivations, generalized derivations, generalized *(α, β)*-derivations and multiplicative (generalized*)-(α, β)*-derivation which satisfy appropriate algebraic conditions on appropriate subsets of the rings. In this article, we also have studied some algebraic identities with multiplicative (generalized*)-(α, β)*-derivation on subsets of rings and we have found conditions on the maps.

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