Reformulating and Strengthening the Theoretical Foundations of Semi-structured Complex Numbers

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Abstract: Recently a new number p (called the unit unstructured number) was invented to solve "division by zero". "p" was used to create a new number set called the semi-structured complex numbers and a new coordinate plane called the real-unstructured xz-plane. However, the definition of p led to a number set that had a poor theoretical foundation and lacked consistency in simple algebraic calculations producing contradictory results when dividing by zero. To overcome these issues, this research redefines the unit p and, in the process, shows that: 1) the new definition of p produces a more theoretically grounded number set that forms a "field" and behaves consistently in algebraic calculations; 2) p can be used to find a viable solution to the logarithm of zero; 3) the new definition of p helps provide an unambiguous understanding of what $\frac{0}{0}=k$ means (where k is any real number). The research proves that $\frac{0}{0}=k$ simply represents 90° clockwise rotation of the vector kp from the positive unstructured z-axis to k on the positive real x-axis along the real-unstructured xz-plane. These results serve to provide a firm foundation for Semi-structured Complex Numbers and support their practical use.

Key words: Division by zero, semi-structured complex numbers, singular matrices.

1. Introduction

1.1. Semi-structured Complex Numbers: An Attempt at Solving Division by Zero

The problem of division by zero is a 12000-year-old problem. Recently there has been a range of research involving division by zero. This includes teaching children on an elementary level possible means and solutions to division by zero and using division by zero to explain cancer development on the most fundamental level. Table A1 in Appendix 1 shows some examples research conducted from 2018 to 2022 involving division by zero.

The problem of division by zero can simply be stated as: What is $\frac{a}{0}$ where "a" is any complex number. There have been several solutions to the problem of division by zero the most recent being the invention of semi-structured complex number set [1]. Notwithstanding, whilst the number set had some merits it was considered to have a poor theoretical foundation and did not quite provide a quality solution to division by zero. Here the theory of semi-structured complex numbers is provided, and the weaknesses of this number set is expounded on.

1.2. Definition of Semi-structured Complex Numbers

Semi-structured complex number set is a set that can be represented by points in a 3-dimensional Euclidean xyz-space and is defined as follows:

A semi-structured complex number is a three-dimensional number of the general form $h = x + yi + zp$; *that is, a linear combination of real* (1), *imaginary* (i) *and unstructured* (p) *units whose coefficients* x, y, z *are real numbers.*

The number h is called semi-structured complex because it contains a structured complex part $(x + yi)$ and an unstructured part (zp). In paper [1], p was defined using Equation (1):

$$p^b = i^{b-f^b(0)}.f^b(1) \tag{1}$$

where

$f^b(c)$	Is a composite function such that $f(c) = 1 - c$
i	Imaginary unit
p	Unstructured unit
b	An exponent belonging to the real number set.

According to authors of [1], powers of "p" have the following values:

Table 1. Powers of p and Associated Values

Powers of $\boldsymbol{p}$	Value
$\boldsymbol{p^1}$	$\mathbf{0}$
$\boldsymbol{p^2}$	$\mathbf{-1}$
$\boldsymbol{p^3}$	$\mathbf{-0}$
$\boldsymbol{p^4}$	$\mathbf{1}$

1.3. Criticisms and Weaknesses of the Theory of Semi-structured Complex Numbers

Table 2 provides seven major weaknesses of the theory of semi-structured complex numbers and reasons why it is not a complete solution to division by zero.

$$ip = \sqrt{-1} \times p = \sqrt{-1 \times p^2}$$

$$ip = \sqrt{-1 \times -1} \quad (\text{Since } \ p^2 = -1 \ \ \text{Table 1}) \tag{2}$$

$$ip = 1$$

The criticisms given in Table 2 can easily be resolved if a new definition for the unstructured unit p is conceived. A new definition of p would lead to an entirely new number system being developed. It must be shown that that number system can be conveniently represented graphically and can be used algebraically alongside all the other number systems that currently exist. To do this, there is a bit of trial and error that must take place. However, to keep things simple initially some assumptions will be made and justified as the mathematics necessary to enable division by zero is developed.

Table 2. Criticisms and Weaknesses of the Theory of Semi-structured Complex Numbers

Criticisms and weakness	Explanation
1. The value of p does not adequately solve the problem $0 \times \frac{1}{0} = \frac{0}{0}$	Given that $p = 0$, this implies that $\frac{0}{0} = \frac{p}{p} = 1$. However, 1 is not the only valid solution to $\frac{0}{0}$. Hence the definition of p does not provide a complete solution to the problem.
2. p is simply a scalar multiple of i.	Definition of p as shown in Equation (1) is based on the complex number i making p simply a scalar multiple of i. This means p and i are not linearly independent and h in Equation $h = x + yi + zp$ a complex number which can be written as $h = x + (y + z)i$. Hence nothing new was discovered.
3. The definition of p leads to the contradiction $ip = 0$ and $ip = 1$.	The original paper [1] asserted that $ip = 0$ (claiming that this is true since p and i are perpendicular vectors). But the definition of p (Equation (1)) and the definition of the complex unit i, implies that $ip = 0$ is a contradiction because of Equation (2).
4. No practical application of semi-structured complex numbers was given.	Other solutions to division by zero provided practical applications. For the theory of semi-structured complex numbers to have a firm foundation, practical examples of the use of these numbers should have been given.
5. It is unclear the link between theoretical construction of natural numbers and the definition of p	The theoretical construction of natural numbers quoted in paper [1] is one of many constructions and the use of this construction was in defining p was not adequately justified.
6. Paper [1] claimed that semi-structured complex numbers is a "field" but does not define the field of semi-structured complex numbers	Paper [1] claimed that this number set is an "ordered field" without first defining the field of semi-structured complex numbers. If there is no definition for the field of semi-structured complex numbers, then how can the axioms be used to prove that the number set is a field. Paper [1] also takes it as axiomatic that a number system that allows division by zero must satisfy the field axioms but other authors who have done work on "division by zero" do not accept this axiom.
7. Paper [1] never proved that semi-structured complex numbers is an ordered field.	The phrase "ordered field" implies that operations such as greater than or less than can be applied to objects within the field. The authors never showed that such operations can be applied to the set of structured complex numbers.

1.4. Major Contributions of Paper

To reiterate, two major problems exist with the "division by zero" problem as presented in this paper:

1. The problem of "division by zero" has not yet been adequately solved.
2. Whilst semi-structured complex number set has some merits, it was considered to have a poor theoretical foundation and did not quite provide a quality solution to "division by zero". This is largely because the unstructured unit p is poorly defined leading to contradictions.

Given these issues and potential implication of solving them, the purpose of this paper was to:

Redefine the unstructured unit p and in the process repurpose semi-structured numbers to adequately solve the division by zero problem.

In the process of achieving this purpose the following major contributions are made by this paper:

1. The unstructured number p was redefined as $p^1 = \frac{1}{0}$. With this new definition, powers of p is represented by the equation:

$$p^n = \frac{\sqrt{2} \times cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)}{f^n(1)} \tag{3}$$

where $f^n(c)$ is a composite function such that $f(c) = 1 - c$

2. The field of Semi-structured Complex Numbers was defined, and proof was given that this field obeys the field axioms.
3. The unit p was used to find a viable solution to the logarithm of zero. The logarithm of zero was

found to be:

$$\log 0 = -p\left(\frac{\pi}{2} + 2k\pi\right) \tag{4}$$

where k is some integer value.

4. The new definition of p helps provide an unambiguous understanding that $\frac{0}{0} = k$ simply represents 90° clockwise rotation of the vector kp from the positive unstructured z-axis to k on the positive real x-axis along the real-unstructured xz-plane.
5. Practical examples of the use of Semi-structured Complex Numbers in solving "division by zero" problems (with the new definition of p) is also given. These examples included representing divergent infinite series as semi-structured complex vectors and making singular matrices invertible.

The rest of this paper is devoted to providing a detailed explanation of how the objective of this paper was achieved and how this in turn gave rise to the development of the major contributions outlined in this paper.

2. Developing a Number to Represent $\frac{1}{0}$

One of the major issues in mathematics is that $\frac{1}{0}$ is considered undefined. As a starting point we need to give a definition to $\frac{1}{0}$. To begin, suppose there a unit number p such that the following assumptions hold:

1. p does not belong to the set of complex numbers $\mathbb{C}$; that is $p \notin \mathbb{C}$.
2. It is assumed that the unit number p belongs to a higher order number set $\mathbb{H}$ called the set of semi-structured complex numbers such that the set of complex numbers is a subset of $\mathbb{H}$. That is $\mathbb{C} \subset \mathbb{H}$.
3. As shown in Table 1, powers of p have the following characteristics:

$$p^1 = \frac{1}{0} \qquad p^2 = -1 \qquad p^3 = -p \qquad p^4 = 1 \tag{5}$$

4. It is also assumed that the values for the powers of p given in Equation (5) repeat for higher power of p. That is

$$p^5 = \frac{1}{0} \qquad p^6 = -1 \qquad p^7 = -p \qquad \ldots \tag{6}$$

Each of these assumptions needs to be justified and their importance outlined. For the purposes of this manuscript p will be referred to as the "unstructured unit". Unstructured here means that p has a structure (or value) $\frac{1}{0}$ that cannot be found in the complex number set that is $\mathbb{C}$. For this reason, assumption 1 must be made; that is $p \notin \mathbb{C}$.

If p does not belong to the set of complex numbers, and p exist (because $\frac{1}{0}$ exist) then p must belong to a larger set of numbers that includes the complex number set. This set is called the set of semi-structured complex numbers represented by the symbol $\mathbb{H}$. The name "semi-structured complex" implies that the set $\mathbb{H}$ contains structured values (that is values that can be found in the set of complex numbers) and unstructured values (that is values that cannot be found in the set of complex numbers). This can be summarized in assumption 2.

Powers of p defined in assumption 3 and assumption 4 are necessary particularly for representing p graphically. However, what is important as this point is to derive a formula for p that would permit the powers of p to have the values that are shown in Equation (5) and Equation (6). This formula is given in Equation (7). The derivation of the formula given in Equation (7) is given in Appendix 1.

$$p^n = \frac{\sqrt{2} \times cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)}{f^n(1)} \tag{7}$$

where

$f^n(c)$	Is a composite function such that $f(c) = 1 - c$

This new definition of the unstructured unit p implies that p is independent of the imaginary unit i. This resolves the second criticism in Table 2.

3. The Implications of Representing the Unstructured Number $a + bp$ Graphically

3.1. The Graphical Representation of the Unstructured Number $a + bp$

The unstructured number $a + bp$ (where a and b are real numbers and p is the unstructured unit) can be represented graphically using the real-unstructured plane (*xz*-plane) as shown in Fig. 1.

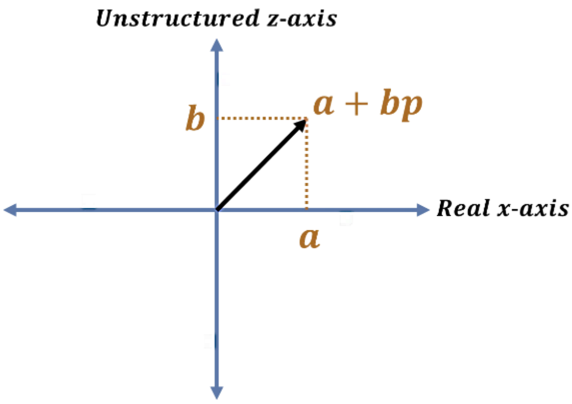

Fig. 1. Unstructured number $a + bp$ represented on real-unstructured plane (*xz*-plane).

Interestingly, if the angle to the *x*-axis is considered then the unstructured number $a + bp$ can be rewritten as shown in Fig. 2.

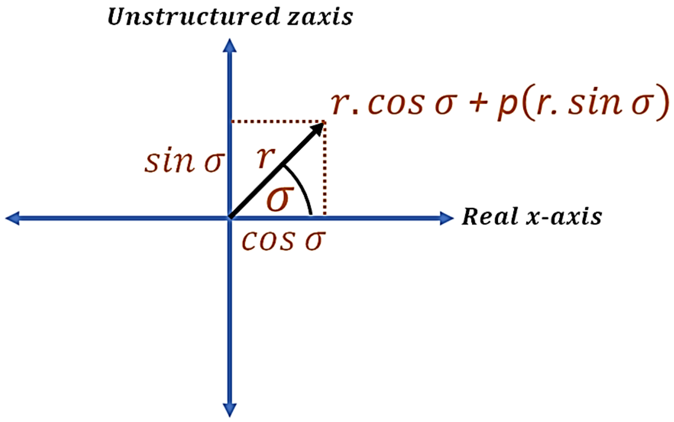

Fig. 2. Unstructured number $a + bp$ represented on real-unstructured plane (*xz*-plane).

From Fig. 2 the first equation for the unstructured number $a + bp$ can be derived. This is given in Equation (8).

$$a + bp = r.(\cos\sigma + p.\sin\sigma) = r.e^{p\sigma} \tag{8}$$

where r is the modulus of the unstructured number $a + bp$ calculated from the equation $r = \sqrt{a^2 + b^2}$. Equation (8) is the polar representation of the unstructured number $a + bp$. Note that Equation (8) is analogous to the polar representation of complex numbers, that is $r.e^{i\theta} = r.(\cos\theta + i.\sin\theta)$.

3.2. Polar Representation of the Unstructured Number $a + bp$ as a Useful Mathematical Tool

Equation (8) is a useful mathematical tool enabling all powers of p to be found. For example, an analogous De Moivre's formula can be applied to Equation (8) to produce Equation (9) that enables other powers of the unstructured unit p to be found.

$$p^n = e^{p\left(\frac{\pi}{2}n\right)} = \cos\left(\frac{\pi}{2}n\right) + p.\sin\left(\frac{\pi}{2}n\right) \tag{9}$$

Equation (8) is also useful in enabling other mathematical operations such a logarithm to be performed on p. For example, Equation (8) can be used to evaluate the logarithm of the unstructured unit p and the logarithm of zero. The logarithm of the unstructured unit p and the logarithm of zero is given in Equation (10) and Equation (11) respectively.

$$\log p = p\left(\frac{\pi}{2} + 2k\pi\right) \tag{10}$$

$$\log 0 = -p\left(\frac{\pi}{2} + 2k\pi\right) \tag{11}$$

where k is some integer value. Proof of Equation (10) and Equation (11) is given in Appendix 3.

3.3. Rotation of the Unstructured Unit p About the Real-Unstructured Plane (xz-Plane)

The powers of the unstructured unit p as calculated from Equation (9), permits rotations of that unit about the real-unstructured plane (xz-plane). The clockwise rotation of the unstructured unit p along the xz-plane is shown in Fig. 3.

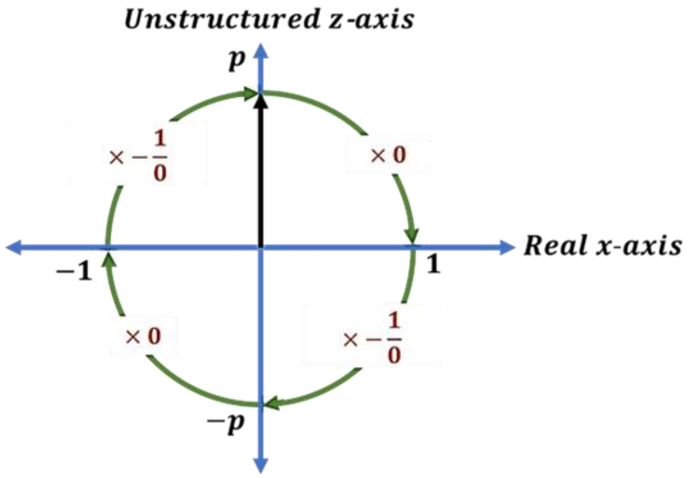

Fig. 3. Clockwise rotation of the unstructured unit p along the real-unstructured plane (xz-plane).

According to Fig. 3, multiplying the unstructured unit p by zero is equivalent to rotating p by an angle of 90° clockwise to produce a value of 1 on the real x-axis. Hence, Fig. 3 implies that multiplication by zero has a significant meaning, a 90° clockwise rotation from the positive unstructured z-axis to the positive real x-axis. From observing Fig. 3, it follows then that Equation (12) is correct.

$$0p = 1 \tag{12}$$

Consequently, it follows from Equation (12) that for any unstructured vector kp, where k is a real number and represents the magnitude of the vector, Equation (13) holds.

$$0(kp) = k \tag{13}$$

This is a very substantial result because it explains why $\frac{0}{0} = k$ for all $k \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The explanation is as follows:

Since $0 \times (kp)$ produces a clockwise rotation from kp to k, it follows that

$$0(kp) = k \qquad \forall k \in \mathbb{R}$$

But since $p = \frac{1}{0}$, then $0(kp) = k$ can be written algebraically as:

$$0\left(k\frac{1}{0}\right) = k \qquad \forall k \in \mathbb{R}$$

Expanding gives:

$$0 \times k \times \frac{1}{0} = k \qquad \forall k \in \mathbb{R}$$

$$\frac{0 \times k \times 1}{0} = k \qquad \forall k \in \mathbb{R}$$

$$\frac{0}{0} = k \qquad \forall k \in \mathbb{R} \tag{14}$$

Hence, from Equation (14) is follows that $\frac{0}{0} = k$ masks the fact that a 90° clockwise rotation of the vector kp is taking place from the positive unstructured z-axis to the positive real x-axis. Equation (14) makes it appear that from a purely algebraic standpoint $\frac{0}{0} = k$ for some real number k. However, rotation about the real-unstructured plane as described here provides a very clear unambiguous understanding of what $\frac{0}{0} = k$ means. $\frac{0}{0} = k$, is simply a 90° clockwise rotation of the vector kp is taking place from the positive unstructured z-axis to k on the positive real x-axis. This is shown in Fig. 4.

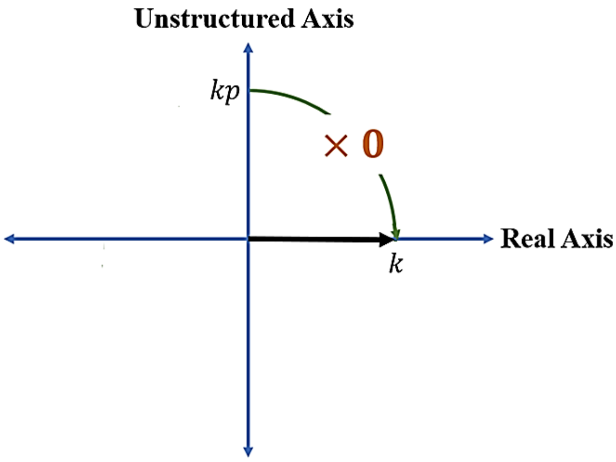

Fig. 4. Meaning of $\frac{0}{0} = k$, a 90° clockwise rotation of the vector kp is taking place from the positive unstructured z-axis to k on the positive real x-axis.

Another important point to note from Equation (13) is that when k = 0, this produces Equation (15).

$$0(0p) = 0 \tag{15}$$

Equation (15) simply implies that the unstructured unit multiplied by zero twice produces a result of zero.

4. Representing Semi-structured Complex Number $h = x + iy + pz$ Graphically

Another interesting point is since they share the real-axis, the real-unstructured plane (xz-plane) can be combined with the real-imaginary plane (xy-plane) to form a 3-dimensional coordinate system that can be used to graph semi-structured complex numbers. This is shown in Fig. 5.

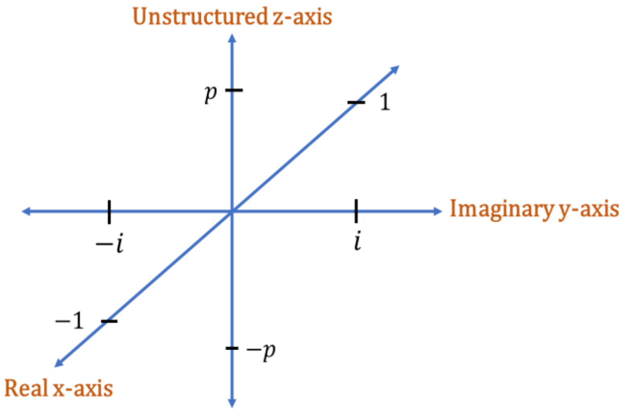

Fig. 5. 3- dimensional coordinate system used to graph semi-structured complex numbers.

The real, imaginary, and unstructured axis are all perpendicular to each other. The semi-structured complex number h can be represented as a vector in this 3- dimensional coordinate system as shown in Fig. 6.

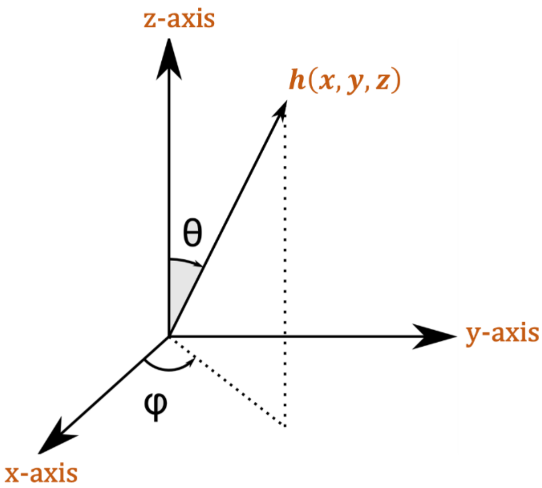

Fig. 6. Semi-structured complex number h.

The semi-structured complex number h can simply be written as the sum of its individual components along the x, y and z-axis. This is given in Equation (16).

$$h = x + yi + zp \tag{16}$$

Here x, y, z are real numbers and i is the imaginary unit whilst p is the unstructured unit. The components x, y, z can be written in terms of θ (the angle h makes with the z-axis) and φ (the angle the projection of h makes with the x-axis). This is given in Equation (17) and Equation (19).

$$x = r \sin\theta \cos\varphi \tag{17}$$

$$y = r \sin\theta \sin\varphi \tag{18}$$

$$z = r\cos\theta \tag{19}$$

where r is the magnitude of the semi-structured number h and is given by $r = \sqrt{x^2 + y^2 + z^2}$.

5. Adding and Multiplying Semi-structured Complex Numbers

Now that is has been shown that these numbers can be represented graphically, to develop the theory of semi-structured complex numbers further it must be shown that these numbers can be added and multiplied; that is the semi-structured complex number set $\mathbb{H}$ is closed under addition and multiplication.

5.1. Addition of Semi-structured Complex Numbers

Consider $h_1 = (x_1 + y_1 i + \mathrm{z}_1\mathrm{p}) \in \mathbb{H}$ *and* $h_2 = (x_2 + y_2 i + \mathrm{z}_2\mathrm{p}) \in \mathbb{H}$.

$$(x_1 + y_1 i + \mathrm{z}_1\mathrm{p}) + (x_2 + y_2 i + \mathrm{z}_2\mathrm{p}) = (x_1 + x_2) + (y_1 + y_2)i + (\mathrm{z}_1 + \mathrm{z}_2)\mathrm{p}$$

Since x_1, x_2, y_1, y_2, z_1 *and* z_2 are all real numbers, then $(x_1 + x_2) + (y_1 + y_2)i + (u_1 + u_2)\mathrm{p} \in \mathbb{H}$. Hence, the semi-structured complex number set $\mathbb{H}$ is closed under addition; that is, $\forall\, h_1 \in \mathbb{H}$ *and* $\forall\, h_2 \in \mathbb{H}$, $h_1 + h_2 \in \mathbb{H}$.

5.2. Multiplication of Semi-structured Complex Numbers

Consider $h_1 = (x_1 + y_1 i + \mathrm{z}_1\mathrm{p}) \in \mathbb{H}$ *and* $h_2 = (x_2 + y_2 i + \mathrm{z}_2\mathrm{p}) \in \mathbb{H}$.

$$h_1\, h_2 = (x_1 + y_1 i + \mathrm{z}_1\mathrm{p}) \times (x_2 + y_2 i + \mathrm{z}_2\mathrm{p})$$

$$h_1\, h_2 = x_1 \times\ (x_2 + y_2 i + \mathrm{z}_2\mathrm{p}) + y_1 i \times\ (x_2 + y_2 i + \mathrm{z}_2\mathrm{p}) + \mathrm{z}_1\mathrm{p} \times\ (x_2 + y_2 i + \mathrm{z}_2\mathrm{p})$$

$$h_1\, h_2 = x_1x_2 + x_1y_2 i + x_1\mathrm{z}_2\mathrm{p} + y_1x_2 i + y_1y_2 ii + y_1\mathrm{z}_2\mathrm{ip} + \mathrm{z}_1x_2\mathrm{p} + \mathrm{z}_1y_2 ip + \mathrm{z}_1\mathrm{z}_2\mathrm{pp}$$

Now $ii = i^2 = -1$; and $\mathrm{pp} = p^2 = -1$.

$$h_1\, h_2 = x_1x_2 + x_1y_2 i + x_1\mathrm{z}_2\mathrm{p} + y_1x_2 i - y_1y_2 + y_1\mathrm{z}_2\mathrm{ip} + \mathrm{z}_1x_2\mathrm{p} + \mathrm{z}_1y_2 ip - \mathrm{z}_1\mathrm{z}_2$$

$$h_1\, h_2 = (x_1x_2 - y_1y_2 - z_1z_2) + (x_1y_2 + y_1x_2)i + (x_1z_2 + z_1x_2)p + (y_1z_2 + z_1y_2)ip$$

Now the unit ip needs to be resolved since there is no axis that represents this component. This can easily be done as follows:

Let

$$(x_1x_2 - y_1y_2 - z_1z_2) + (x_1y_2 + y_1x_2)i = [A + Bi]$$

where $A = (x_1x_2 - y_1y_2 - z_1z_2)$ and $B = (x_1y_2 + y_1x_2)$

Additionally,

$$(x_1z_2 + z_1x_2)p + (y_1z_2 + z_1y_2)ip = [(x_1z_2 + z_1x_2) + (y_1z_2 + z_1y_2)i]p$$

Let

$$[(x_1z_2 + z_1x_2) + (y_1z_2 + z_1y_2)i]p = [C + Di]p$$

where $C = (x_1z_2 + z_1x_2)$ and $D = (y_1z_2 + z_1y_2)$

Hence

$$h_1\, h_2 = (x_1x_2 - y_1y_2 - z_1z_2) + (x_1y_2 + y_1x_2)i + (x_1z_2 + z_1x_2)p + (y_1z_2 + z_1y_2)ip$$

$$h_1\, h_2 = [A + Bi] + [C + Di]p$$

Equation $h_1\, h_2 = [A + Bi] + [C + Di]p$ can be written in exponential form as follows:

$$h_1\, h_2 = r\left(\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}\alpha} + \mathrm{r}_\theta \mathrm{e}^{\mathrm{i}\theta}\mathrm{p}\right)$$

where $\mathrm{r.r}_\alpha \mathrm{e}^{\mathrm{i}\alpha} = A + Bi$ and $\mathrm{r.r}_\theta \mathrm{e}^{\mathrm{i}\theta} = C + Di$. Additionally, $|\mathrm{r}_\alpha|^2 + |\mathrm{r}_\theta|^2 = 1$.

We can multiply equation $h_1\, h_2 = r\left(\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}\alpha} + \mathrm{r}_\theta \mathrm{e}^{\mathrm{i}\theta}\mathrm{p}\right)$ by $\mathrm{e}^{-\mathrm{i}\theta}$. By doing this, the properties of $h_1\, h_2$ such as the absolute value of $h_1\, h_2$ remains unchanged. Hence:

$$h_1\, h_2 = r\left(\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}\alpha} + \mathrm{r}_\theta \mathrm{e}^{\mathrm{i}\theta}\mathrm{p}\right) \times \mathrm{e}^{-\mathrm{i}\theta}$$

$$h_1\, h_2 = r\left(\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}(\alpha-\theta)} + \mathrm{r}_\theta \mathrm{e}^{\mathrm{i}(\theta-\theta)}\mathrm{p}\right)$$

$$h_1\, h_2 = r\left(\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}\varphi} + \mathrm{r}_\theta \mathrm{p}\right)$$

Since $|\mathrm{r}_\alpha|^2 + |\mathrm{r}_\theta|^2 = 1$, let $\mathrm{r}_\alpha = \sin\theta$ and $\mathrm{r}_\theta = \cos\theta$. Additionally, $e^{i\varphi} = cos\,\varphi + i.sin\,\varphi$. This gives:

$$h_1\, h_2 = r(\sin\theta \times (cos\,\varphi + i.sin\,\varphi) + (\cos\theta)\mathrm{p})$$

$$h_1\, h_2 = r\sin\theta\, cos\,\varphi + r.i\sin\theta\, sin\,\varphi + r.p\cos\theta$$

But substituting Equation (17) and Equation (19) into this result gives:

$$h_1\, h_2 = x + yi + zp$$

Consequently, for the semi-structured complex number set $\mathbb{H}$, the operation of multiplication is closed, that is $\forall\, h_1 \in \mathbb{H}, \forall\, h_2 \in \mathbb{H},\ h_1\, h_2 \in \mathbb{H}$.

6. Defining the Field of Semi-structured Complex Numbers

One of the major issues with the original paper on semi-structured complex numbers is that the field of semi-structured complex numbers was never defined. Having just shown that the operations of addition "+" and multiplication "×" can be performed on semi-structured complex numbers (that is, the set $\mathbb{H}$ is closed under the operations of addition "+" and multiplication "×"), this paper defines the field of semi-structured complex numbers. Formally, the system of semi-structured complex numbers consists of the set:

$$\mathbb{H} = \{h = x + yi + zp\ |\text{where } x, y \text{ and } z \text{ are real numbers}\} \tag{20}$$

The field of semi-structured complex numbers is defined as follows:

The field of semi-structured complex numbers is the set of elements $\mathbb{H}$ with two operations $"+"$ *and* $"."$ such that:

1. $\langle \mathbb{H}, + \rangle$ *is a commutative group*

2. $\langle \mathbb{H}^{\times}, . \rangle$ *is a commutative group* for non-zero elements of $\mathbb{S}$
3. Addition $(+)$ and multiplication $(.)$ are linked by the distributive property; that is, for all $h_1, h_2, h_3 \in \mathbb{H}$,

$$h_1.(h_2 + h_3) = h_1.h_2 + h_1.h_3 \text{ and}$$
$$(h_2 + h_3).h_1 = h_2.h_1 + h_3.h_1$$

The addition and multiplication for semi-structured complex numbers are defined in such a way that the rules of addition and multiplication are consistent with the rules for real numbers.

To show that the semi-structured complex numbers set forms a field, these numbers must satisfy the field axioms provided in Appendix 4. The proof that these numbers satisfy the field axioms are given in Appendix 5. In satisfying the field axioms the authors of this paper have legitimately proven that semi-structured complex numbers do in fact exist and agree with the normal rules of algebra. It is not axiomatic that any number set that solves the division by zero problem must satisfy the field axioms. Nevertheless, if such a number set does satisfy the field axioms, then 1) the number set can easily be used in everyday algebraic expressions and be incorporated into algebraic problems, 2) the number set can be used to form more complicated structures such as vector spaces and hence solve more complex problems that may involve "division by zero".

It is also worth noting that the semi-structured complex number set $\mathbb{H}$ does not form an ordered field. For the objects in a field to have an order, operations such as greater than or less than can be applied to these objects. In the case of semi-structured complex numbers, it is possible to define a partial order using for example the following rule:

$$x_1 + y_1 i + z_1 p < x_2 + y_2 i + z_2 p$$
$$\text{if and only if } x_1 < x_2,\ y_1 < y_2 \text{ and } z_1 < z_2$$

However, it is impossible to define a total order on the set in such a way that it becomes an "ordered field". This is because in an ordered field the square of any non-zero number is greater than 0; this is not the case with semi-structured complex numbers.

7. Practical Applications of Semi-structured Complex Numbers

Having fully defined the Semi-structured Complex Number set, for brevity two practical applications of the use of these numbers were considered and presented here.

7.1. Representation of the Power Series Expansion of $\frac{1}{1-x}$ When $x = 1$

The Maclaurin series expansion for $\frac{1}{1-x}$ is the geometric series given by Equation (21).

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{21}$$

When $x = 1$, Equation (21) results in Equation (22).

$$\begin{aligned} \frac{1}{1-1} &= 1 + 1 + 1^2 + 1^3 + \cdots \\ \frac{1}{0} &= 1 + 1 + 1 + 1 + \cdots \end{aligned} \tag{22}$$

Equation (22) can easily be presented by a semi-structured complex number. This is shown in Equation (23).

$$p = 1 + 1 + 1 + 1 + \cdots \tag{23}$$

At $x = 1$, the Maclaurin series expansion for $\frac{1}{1-x}$ is outside its range of applicability. This means that the assumptions made by the expansion have gone as far as the mathematics can take it and the expansion cannot be interpreted at $x = 1$ (Equation (22)). However, with the application of semi-structured complex numbers, the results shown in Equation (23) can be interpreted. Equation (23) says that the divergent sum can be represented as a unit vector p. This is a very handy tool for solving problems involving divergent series (the full implication of this is outside the scope of this paper). Nevertheless, it is worth noting that this is an area of mathematics currently unexplored.

7.2. Making a Singular Matrix Invertible

The second example of the use of semi-structured complex numbers can be seen its application to matrices. Matrices are mathematical objects used extensively in mathematics, engineering, and science. In linear transformations, matrices act 'scale' vectors and or change their direction.

Suppose a $n \times n$ matrix A represents a linear transformation on all vectors in an n-dimensional vector space. An n-dimensional vector space is simply a set of vectors with n elements (arranged in a row or column). The absolute value of the determinant of matrix A would represent the scale factor by which a region of space occupied by the n-dimensional vector space would increase or decrease. The sign of the determinant of matrix A indicates whether the orientation of the space changes (or flips). If the determinant of matrix A is zero, then it implies that the space occupied by the n-dimensional vectors would be diminished (squished) to a lower dimension. This implies that, the n-dimensional space would become an m dimensional space where $m < n$. Unfortunately, once this dimension reduction effect occurs, information of the original shape is lost.

Matrices with a determinant of zero are called singular matrices and such matrices have no inverses. The inverse of a matrix is usually used to recover information or reverse the transformation that a matrix originally caused. Without an inverse, singular matrices used in transformations can cause a loss of information.

However, with the invention of semi-structured complex numbers, it is possible to determine the inverse of a singular matrix. With the use of semi-structured complex numbers, given a singular matrix A, then A^{-1} can be found using Equation (24):

$$A^{-1} = p \times adj(A) \tag{24}$$

Here p is the semi-structured rotational unit. A non-rigorous proof of Equation (24) is given in Table 3.

Table 3. Proof of Inverse of Singular Matrix

Let A be a singular matrix so that $|\mathrm{A}| = 0$. (Note $|\mathrm{A}|$ is the determinant of A)

Now:

$$I = A^{-1} \times A$$

$$I = A^{-1} \times A = \left(\frac{1}{|\mathrm{A}|} \times adj(A)\right) \times A$$

Since $|\mathrm{A}| = 0$

$$I = A^{-1} \times A = \left(\frac{1}{0} \times adj(A)\right) \times A$$

Hence

$$I = A^{-1} \times A = p \times adj(A) \times A$$

$$A^{-1} = p \times adj(A)$$

An example of the use of Equation (24) is given for a singular 2×2 matrix $B = \begin{pmatrix} b & b \\ c & c \end{pmatrix}$ in Table 4.

Table 4. Proof of Inverse of Singular Matrix for 2×2 Matrices

Suppose B is a singular matrix such that $B = \begin{pmatrix} b & b \\ c & c \end{pmatrix}$

Then the inverse of B is: $B^{-1} = \frac{1}{bc-bc}\begin{pmatrix} c & -b \\ -c & b \end{pmatrix} = \frac{1}{0}\begin{pmatrix} c & -b \\ -c & b \end{pmatrix} = p\begin{pmatrix} c & -b \\ -c & b \end{pmatrix}$

(where p is the unstructured unit).

PROOF:

Hence $B \times B^{-1}$ gives:

$$
\begin{aligned}
B \times B^{-1} &= \begin{pmatrix} b & b \\ c & c \end{pmatrix} \times \frac{1}{bc-bc}\begin{pmatrix} c & -b \\ -c & b \end{pmatrix} \\
B \times B^{-1} &= \frac{1}{bc-bc} \times \begin{pmatrix} b & b \\ c & c \end{pmatrix} \times \begin{pmatrix} c & -b \\ -c & b \end{pmatrix} \\
B \times B^{-1} &= \frac{1}{bc-bc} \times \begin{pmatrix} bc-bc & b^2-b^2 \\ c^2-c^2 & bc-bc \end{pmatrix} \\
B \times B^{-1} &= \begin{bmatrix} \left(\frac{bc-bc}{bc-bc}\right) & \left(\frac{b^2-b^2}{bc-bc}\right) \\ \left(\frac{c^2-c^2}{bc-bc}\right) & \left(\frac{bc-bc}{bc-bc}\right) \end{bmatrix}
\end{aligned} \tag{25}
$$

The following three expressions must be noted:

$bc - bc = \frac{1}{2}(b-b)(c-c) = \frac{1}{2}(0)(0) = \frac{1}{2}(-p)(-p) = \frac{1}{2}(p^2) = \frac{1}{2}(-1) = -\frac{1}{2} = -0.5$ (since $0 = -p$)

$b^2 - b^2 = (b-b)(b+b) = (0)(2b) = 0b$

$c^2 - c^2 = (c-c)(c+c) = (0)(2c) = 0c$

Substituting these expressions into $B \times B^{-1}$ gives

$$
\begin{aligned}
B \times B^{-1} &= \begin{bmatrix} \left(\frac{bc-bc}{bc-bc}\right) & \left(\frac{b^2-b^2}{bc-bc}\right) \\ \left(\frac{c^2-c^2}{bc-bc}\right) & \left(\frac{bc-bc}{bc-bc}\right) \end{bmatrix} \\
B \times B^{-1} &= \begin{bmatrix} \left(\frac{-0.5}{-0.5}\right) & \left(\frac{0b}{-0.5}\right) \\ \left(\frac{0c}{-0.5}\right) & \left(\frac{-0.5}{-0.5}\right) \end{bmatrix} \\
B \times B^{-1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned} \tag{26}
$$

Hence using semi-structured complex numbers, the inverse of a 2×2 matrix can be found. This implies that that no information is lost if singular 2×2 matrices are used as transformations.

8. Discussion

Having defined the set and field of semi-structured complex numbers and illustrating the applicability of this set, Table 5 provides a summary of how major improvements to the set of semi-structured complex numbers has been made in this manuscript.

Table 5. Improvements Made to the Theory of Semi-structured Complex Numbers

Weaknesses of Semi-structured complex numbers	**Improvements made with semi-structured complex numbers**
1. The value of p does not adequately solve the problem $0 \times \frac{1}{0}$	The value of p is now defined as $p = \frac{\mathbf{1}}{\mathbf{0}}$. This definition has led to the discovery that $\frac{0}{0} = k$ simply represents 90° clockwise rotation of the vector kp from the positive unstructured z-axis to k on the positive real x-axis along the real-unstructured xz-plane.
2. p is simply a scalar multiple of i.	The unstructured number p is not defined in terms of i but is defined in terms of a periodic cos function as shown in Equation (7).
3. The definition of p leads to a contradiction $ip = 0$ and $ip = 1$.	This contradiction is resolved in section 5.1 by shown that the expression $h = a + bi + cp + dip$ (for real numbers a, b, c, d) is in fact an alternative form of the expression $h = x + yi + zp$ (for real numbers x, y, z); that is, $h = a + bi + cp + dip$ can be written as $h = x + yi + zp$. This fact eliminates the contradiction $ip = 0$ and $ip = 1$.
4. No practical application of semi-structured complex numbers was given.	Practical examples of the use of semi-structured complex numbers are given; particularly expressing divergent series as a simple semi-structured complex vector and finding the inverse of a singular matrix.
5. It is unclear the link between theoretical construction of natural numbers and the definition of p	In this manuscript, the definition of the unstructured number p does not depend on the theoretical construction of natural numbers. Rather it depends on well-established general tools from trigonometry and functions.
6. Paper [1] claimed that semi-structured complex numbers is a "field" but does not define the field of semi-structured complex numbers	The field of semi-structured complex numbers was defined and used to show that the field axioms apply to this number set. This paper does not take it as axiomatic that a number system that allows division by zero must satisfy the field axioms but does illustrate that regular algebraic operations can be done with this number set.
7. Paper [1] never proved that semi-structured numbers is an ordered field.	This research illustrated that it is not possible to define a total order on the set of semi-structured complex numbers in such a way that it becomes an "ordered field". This is because in an ordered field the square of any non-zero number is greater than 0. However, this is not the case with semi-structured complex numbers.

9. Conclusion

Division by zero has been a challenging issue in mathematics for centuries, without any concrete solution to what it means to divide by zero. The aim of this paper redefines the unstructured unit p and in the process repurpose semi-structured numbers to adequately solve the division by zero problem.

The paper shows that p can be defined in terms of the periodic cosine function. This new definition of p produces a more theoretically grounded number set that forms a "field" and behaves consistently in algebraic calculations. Proof was given that this field obeys the field axioms.

The new definition of p also led to a viable solution to the logarithm of $\frac{1}{0}$. The logarithm of 0 was also determined. Practical examples of the utility of Semi-structured Complex Numbers in representing divergent series in mathematics as simple vectors and finding the inverse of a singular matrix.

The new definition also provides an unambiguous understanding of what $\frac{0}{0} = k$ means (where k is any real number). The research proves that $\frac{0}{0} = k$ simply represents 90° clockwise rotation of the vector kp from the positive unstructured z-axis to k the positive real x-axis along the real-unstructured xz-plane.

The successful development of this proposed number set has implications not just in the field of mathematics but in other areas of science where mathematics is frequently used and division by zero is essential.

Appendix

Appendix 1: Research Conducted from 2018 to 2022 Involving Division by Zero

Table A1. Research Conducted on Division by Zero from 2018 to 2022

Research	Research Aim
[2, 3, 4]	Explores the application of division by zero in calculus and differentiation
[5]	Uses classical logic and Boolean algebra to show the problem of division by zero can be solved using today's mathematics
[6]	Develops an analogue to Pappus Chain theorem with Division by Zero
[7]	This paper proposes that the quantum computation being performed by the cancer cell at its most fundamental level is the division by zero. This is the reason for the insane multiplication of cancer cells at its most fundamental scale.
[8]	Explores evidence to suggest zero does divide zero
[9]	Considered using division by zero to compare incomparable abstract objects taken from two distinct algebraic spaces
[10]	Show recent attempts to divide by zero
[11]	Generalize a problem involving four circles and a triangle and consider some limiting cases of the problem by division by zero.
[12]	Paper considers computing probabilities from zero divided by itself
[13, 14]	Considers how division by zero is taught on an elementary level
[15]	Develops a method to avoid division by zero in Newton's Method
[16]	This work attempts to solve division by zero using a new form of optimization called Different-level quadratic minimization (DLQM)

Appendix 2: Finding a Formula for p^n

Suppose there is a unit number p^n such that:

$$p^1 = \frac{1}{0} \qquad p^2 = -1 \qquad p^3 = -\frac{1}{0} \qquad p^4 = 1 \tag{27}$$

A formula for the powers of p can be found that considering the pattern of numbers formed by numerator and denominator of powers of p separately as shown in Table A2Table .

Table A2. Pattern of Numbers Formed by Numerator and Denominator of Powers of p

Power of p	1	2	3	4	5	6	7	8	⋯
Numerator of powers of p	1	−1	−1	1	1	−1	−1	1	⋯
Denominator of powers of p	0	1	0	1	0	1	0	1	⋯

If the numerator is considered, the pattern of numbers is $1, -1, -1, 1, -1, -1, 1, \cdots$. This is a repeating (but not alternating) pattern. Additionally, this pattern of numbers for the numerator repeats every four terms.

One of the tricks to representing a repeating sequence is to use a trigonometric function. Since the pattern is repeating this is like moving around in a circle. Hence, consider the unit circle shown in with the coordinates of four points given.

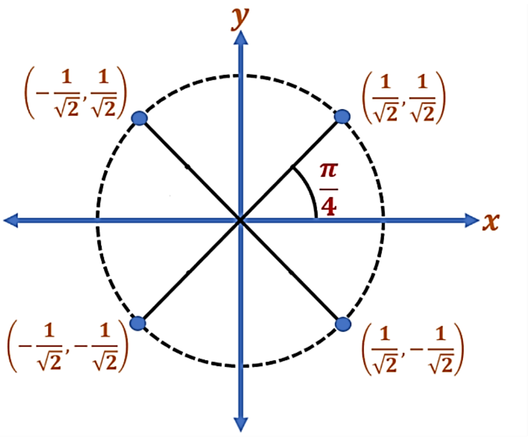

Fig. A1. Unit circle showing four points $\frac{\pi}{2}$ part.

Since there are 4 equidistant points on the circle each point is $\frac{\pi}{2}$ part with the first point being $\frac{\pi}{4}$ off from the positive x axis. If we consider the x-coordinate of each point, we have the sequence (28):

$$\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \cdots \tag{28}$$

This sequence can be represented by the Equation (29):

$$cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right) \tag{29}$$

If sequence (2 is multiplied by $\sqrt{2}$ then this produces pattern (30):

$$1, -1, -1, 1, -1, -1, 1, \cdots \tag{30}$$

This is the exact same pattern as the pattern of numbers for the numerator of p. Hence, the numerator of p can be represented by Equation (31):

$$\sqrt{2} \times cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right) \tag{31}$$

Deriving a formula to represent the denominator required a bit of trial and error. There are several ways to represent the denominator. However, through trial and error the formula $f(c) = 1 - c$ was selected. Putting this function into itself n times with its argument being 1, (that is $f^n(1)$) produces the pattern 0,10,1,0,1,0. This is the exact same pattern as the pattern of numbers for the denominator of p. Hence combining the formula for the numerator and the denominator gives Equation (7).

Appendix 3: Proof of the $log\, p$ and the $log\, 0$

From Equation (8).

$$a + bp = r.e^{p\sigma} \tag{32}$$

Taking the log of both sides gives:

$$\ln(a + bp) = \ln(r.e^{p\sigma})$$
$$\ln(a + bp) = \ln(r) + \ln(e^{p\sigma})$$
$$\ln(a + bp) = \ln(r) + p\sigma \tag{33}$$

When a = 0 and b = 1, this implies r = 1. Hence from Equation (33):

$$\ln(p) = \ln(1) + p\left(\frac{\pi}{2}\right)$$

$$\ln(p) = p\left(\frac{\pi}{2}\right) \tag{34}$$

Since $\cos\sigma$ and $\sin\sigma$ are periodic, this implies that their values repeat every $2\pi k$ for some integer k. Hence:

$$\ln(p) = \ln(1) + p\left(\frac{\pi}{2} + 2\pi k\right)$$

$$\ln(p) = p\left(\frac{\pi}{2} + 2\pi k\right) \tag{35}$$

Additionally, since $p = \frac{1}{0}$, this implies:

$$\ln(p) = p\left(\frac{\pi}{2} + 2\pi k\right)$$

$$\ln(1) - \ln(0) = p\left(\frac{\pi}{2} + 2\pi k\right)$$

$$-\ln(0) = p\left(\frac{\pi}{2} + 2\pi k\right)$$

$$\ln(0) = -p\left(\frac{\pi}{2} + 2\pi k\right) \tag{36}$$

Appendix 4. Field Axioms

Definition: A field is a nonempty set F containing at least 2 elements alongside the two binary operations of addition, $f_{+}: F \times F \to F$ such that $f_{+}(x,y) = x + y$ and multiplication $f_{.}(x,y) = x \cdot y$ that satisfy the following 11 axioms:

1. The operation of addition is closed, that is $\forall x \in F\ and\ \forall y \in F, x + y \in F$.
2. The operation of addition is commutative, that is $\forall x \in F, \forall y \in F, x + y = y + x$.
3. The operation of addition is associative, that $is\ \forall x \in F, \forall y \in F, \forall z \in F, x + (y + z) = (x + y) + z$.
4. The operation of addition has the additive identity element of 0 such that $\forall x \in F, x + 0 = x$
5. The operation of addition has the additive inverse element of $-x$ such that $\forall x \in F, x + (-x) = 0$.
6. The operation of multiplication is closed, that is $\forall x \in F, \forall y \in F, xy \in F$.
7. The operation of multiplication is commutative, that is $\forall x \in F, \forall y \in F, xy = yx$.
8. The operation of multiplication is associative, that is $\forall x \in F, \forall y \in F, \forall z \in F, x(yz) = (xy)z$.
9. The operation of multiplication has the multiplicative identity element of 1 such that $\forall x \in F, 1.x = x$.
10. The operation of multiplication has the multiplicative inverse element of $\frac{1}{x}$ such that $\forall x \in F\ where\ x \neq 0, x \cdot \frac{1}{x} = 1$.
11. The operation of multiplication is distributive over addition, that is, $\forall x \in F, \forall y \in F, \forall z \in F, x(y + z) = xy + xz$.

Appendix 5. Proof that semi-structured complex numbers obey the field axioms

Axiom 1: The operation addition is closed, that is, $\forall\, \boldsymbol{h_1} \in \mathbb{H}\ \boldsymbol{and}\ \forall\, \boldsymbol{h_2} \in \mathbb{H},\ \boldsymbol{h_1} + \boldsymbol{h_2} \in \mathbb{H}$.

Consider $h_1 = (x_1 + y_1 i + \mathrm{z}_1\mathrm{p}) \in \mathbb{H}\ and\ h_2 = (x_2 + y_2 i + \mathrm{z}_2\mathrm{p}) \in \mathbb{H}$.

$(x_1 + y_1 i + \mathrm{z}_1\mathrm{p}) + (x_2 + y_2 i + \mathrm{z}_2\mathrm{p}) = (x_1 + x_2) + (y_1 + y_2)i + (\mathrm{z}_1 + \mathrm{z}_2)\mathrm{p}$

Since x_1, x_2, y_1, y_2, z_1 *and* z_2 are all real numbers, then $(x_1 + x_2) + (y_1 + y_2)i + (u_1 + u_2)\text{p} \in \mathbb{H}$. Hence, the semi-structured complex number set $\mathbb{H}$ is closed under addition; that is, $\forall\, h_1 \in \mathbb{H}$ *and* $\forall\, h_2 \in \mathbb{H}$, $h_1 + h_2 \in \mathbb{H}$.

Axiom 2: The operation of addition is commutative, that is $\forall\, \boldsymbol{h_1} \in \mathbb{H}, \forall\, \boldsymbol{h_2} \in \mathbb{H}$, $\boldsymbol{h_1} + \boldsymbol{h_2} = \boldsymbol{h_2} + \boldsymbol{h_1}$**.**

Consider $h_1 = (x_1 + y_1 i + z_1\text{p}) \in \mathbb{H}$ *and* $h_2 = (x_2 + y_2 i + z_2\text{p}) \in \mathbb{H}$.

$h_1 + h_2 = (x_1 + y_1 i + z_1 p) + (x_2 + y_2 i + z_2 p)$

$h_1 + h_2 = (x_1 + x_2) + (y_1 + y_2)i + (z_1 + z_2)p$

$h_1 + h_2 = (x_2 + x_1) + (y_2 + y_1)i + (z_2 + z_1)p$

$h_1 + h_2 = (x_2 + y_2 i + z_2 p) + (x_1 + y_1 i + z_1 p)$

But

$h_2 + h_1 = (x_2 + y_2 i + z_2 p) + (x_1 + y_1 i + z_1 p)$

Hence

$h_1 + h_2 = h_2 + h_1$

Therefore, since $h_1 + h_2 = h_2 + h_1$. Consequently, semi-structured complex number set $\mathbb{H}$ is commutative under addition.

Axiom 3: The operation of addition is associative, that ***is*** $\forall\, \boldsymbol{h_1} \in \mathbb{H}, \forall\, \boldsymbol{h_2} \in \mathbb{S}, \forall\, \boldsymbol{h_3} \in \mathbb{H}$, $\boldsymbol{h_1} + (\boldsymbol{h_2} + \boldsymbol{h_3}) = (\boldsymbol{h_1} + \boldsymbol{h_2}) + \boldsymbol{h_3}$.

Consider $h_1 = (x_1 + y_1 i + z_1\text{p}) \in \mathbb{H}$; $h_2 = (x_2 + y_2 i + z_2\text{p}) \in \mathbb{H}$ and $h_3 = (x_3 + y_3 i + z_3\text{p}) \in \mathbb{H}$.

$h_1 + (h_2 + h_3) = (x_1 + y_1 i + z_1\text{p}) + [(x_2 + y_2 i + z_2\text{p}) + (x_3 + y_3 i + z_3\text{p})]$

$h_1 + (h_2 + h_3) = (x_1 + y_1 i + z_1\text{p}) + [(x_2 + x_3) + (y_2 + y_3)i + (z_2 + z_3)\text{p}]$

$h_1 + (h_2 + h_3) = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3)i + (z_1 + z_2 + z_3)\text{p}$

$h_1 + (h_2 + h_3) = [(x_1 + x_2) + (y_1 + y_2)i + (z_1 + z_2)\text{p}] + (x_3 + y_3 i + z_3\text{p})$

But

$(h_1 + h_2) + h_3 = [(x_1 + x_2) + (y_1 + y_2)i + (z_1 + z_2)\text{p}] + (x_3 + y_3 i + z_3\text{p})$

Hence $h_1 + (h_2 + h_3) = (h_1 + h_2) + h_3$

Hence, the semi-structured complex number set $\mathbb{H}$ is associative under addition.

Axiom 4: The operation of addition has the additive identity element of 0 such that $\forall \boldsymbol{h} \in \mathbb{H}, \boldsymbol{h} + \boldsymbol{0} = \boldsymbol{h}$**.**

Consider $h_1 = (x_1 + y_1 i + z_1\text{p}) \in \mathbb{H}$ *and* $h_2 = (0 + 0i + 0^2\text{p}) \in \mathbb{H}$.

$h_1 + 0 = (x_1 + y_1 i + z_1\text{p}) + (0 + 0i + 0^2\text{p})$

$h_1 + 0 = (x_1 + 0) + (y_1 + 0)i + (z_1 + 0^2)p$

Since x_1, y_1, z_1 *and* 0 are real numbers, then $x_1 + 0 = x_1$; $y_1 + 0 = y_1$; and $z_1 + 0^2 = z_1$. Hence

$h_1 + 0 = x_1 + y_1 i + z_1\text{p}$

$h_1 + 0 = h_1$

It is important to note that according to Equation (12) $0p$ is 1. Hence, we need to have $0^2p = 0(0p) = 0(1) = 0$. Of course, $0^2 = 0$. However, the term 0^2 is used to emphasize that p is multiplied by zero twice to produce the result of 0.

Consequently, semi-structured complex number set $\mathbb{H}$ has the additive identity element of 0.

Axiom 5: The operation of addition has the additive inverse element of $-\boldsymbol{h}$ such that $\forall \boldsymbol{h} \in \mathbb{H}, \boldsymbol{h} + (-\boldsymbol{h}) = \boldsymbol{0}$.

Consider $h = (x_1 + y_1 i + z_1\mathrm{p}) \in \mathbb{H}\ and\ h = (x_1 + y_1 i + z_1\mathrm{p}) \in \mathbb{H}$.

$h_2 = -(x_1 + y_1 i + z_1\mathrm{p}) = (-x_1 - y_1 i - z_1\mathrm{p})$

$h + (-h) = x_1 + y_1 i + z_1\mathrm{p} - x_1 - y_1 i - z_1\mathrm{p}$

$h + (-h) = (x_1 - x_1) + (y_1 - y_1)i + (z_1 - z_1)p$

$h + (-h) = 0 + 0i + 0p = 0 + 0i + 0^2p = 0$

It is important to note that according to Equation (12) $0p$ is 1. Hence, we need to have $0^2p = 0(0p) = 0(1) = 0$. Of course, $0^2 = 0$. However, the term 0^2 is used to emphasize that p is multiplied by zero twice to produce the result of 0.

Hence, the semi-structured complex number set $\mathbb{H}$ has the additive inverse element of $-h$ such that $\forall h \in \mathbb{H}, h + (-h) = 0$

Axiom 6: The operation of multiplication is closed, that is $\forall\, \boldsymbol{h_1} \in \mathbb{H}, \forall\, \boldsymbol{h_2} \in \mathbb{S}$, $\boldsymbol{h_1}\, \boldsymbol{h_2} \in \mathbb{H}$.

Consider $h_1 = (x_1 + y_1 i + \mathrm{z}_1\mathrm{p}) \in \mathbb{H}\ and\ h_2 = (x_2 + y_2 i + \mathrm{z}_2\mathrm{p}) \in \mathbb{H}$.

$h_1\, h_2 = (x_1 + y_1 i + \mathrm{z}_1\mathrm{p}) \times (x_2 + y_2 i + \mathrm{z}_2\mathrm{p})$

$h_1\, h_2 = x_1 \times (x_2 + y_2 i + \mathrm{z}_2\mathrm{p}) + y_1 i \times (x_2 + y_2 i + \mathrm{z}_2\mathrm{p}) + \mathrm{z}_1\mathrm{p} \times (x_2 + y_2 i + \mathrm{z}_2\mathrm{p})$

$h_1\, h_2 = x_1x_2 + x_1y_2 i + x_1\mathrm{z}_2\mathrm{p} + y_1x_2 i + y_1y_2 ii + y_1\mathrm{z}_2\mathrm{ip} + \mathrm{z}_1x_2\mathrm{p} + \mathrm{z}_1y_2 ip + \mathrm{z}_1\mathrm{z}_2\mathrm{pp}$

Now $ii = i^2 = -1$; and $pp = p^2 = -1$.

$h_1\, h_2 = x_1x_2 + x_1y_2 i + x_1\mathrm{z}_2\mathrm{p} + y_1x_2 i - y_1y_2 + y_1\mathrm{z}_2\mathrm{ip} + \mathrm{z}_1x_2\mathrm{p} + \mathrm{z}_1y_2 ip - \mathrm{z}_1\mathrm{z}_2$

$h_1\, h_2 = (x_1x_2 - y_1y_2 - z_1z_2) + (x_1y_2 + y_1x_2)i + (x_1z_2 + z_1x_2)p + (y_1z_2 + z_1y_2)ip$

Now the unit ip needs to be resolved since there is no axis that represents this component. This can easily be done as follows:

Let

$(x_1x_2 - y_1y_2 - z_1z_2) + (x_1y_2 + y_1x_2)i = [A + Bi]$

where $A = (x_1x_2 - y_1y_2 - z_1z_2)$ and $B = (x_1y_2 + y_1x_2)$

Additionally,

$(x_1z_2 + z_1x_2)p + (y_1z_2 + z_1y_2)ip = [(x_1z_2 + z_1x_2) + (y_1z_2 + z_1y_2)i]p$

Let

$[(x_1z_2 + z_1x_2) + (y_1z_2 + z_1y_2)i]p = [C + Di]p$

where $C = (x_1z_2 + z_1x_2)$ and $D = (y_1z_2 + z_1y_2)$

Hence

$h_1\, h_2 = (x_1x_2 - y_1y_2 - z_1z_2) + (x_1y_2 + y_1x_2)i + (x_1z_2 + z_1x_2)p + (y_1z_2 + z_1y_2)ip$

$h_1\, h_2 = [A + Bi] + [C + Di]p$

Equation $h_1\, h_2 = [A + Bi] + [C + Di]p$ can be written in exponential form as follows:

$h_1\, h_2 = r\left(\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}\alpha} + \mathrm{r}_\theta \mathrm{e}^{\mathrm{i}\theta}\mathrm{p}\right)$

where $\mathrm{r}.\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}\alpha} = A + Bi$ and $\mathrm{r}.\mathrm{r}_\theta \mathrm{e}^{\mathrm{i}\theta} = C + Di$. Additionally, $|\mathrm{r}_\alpha|^2 + |\mathrm{r}_\theta|^2 = 1$.

We can multiply equation $h_1\, h_2 = r\left(\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}\alpha} + \mathrm{r}_\theta \mathrm{e}^{\mathrm{i}\theta}\mathrm{p}\right)$ by $\mathrm{e}^{-\mathrm{i}\theta}$. By doing this, the properties of $h_1\, h_2$ such as the absolute value of $h_1\, h_2$ remains unchanged. Hence:

$h_1\, h_2 = r\left(\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}\alpha} + \mathrm{r}_\theta \mathrm{e}^{\mathrm{i}\theta}\mathrm{p}\right) \times \mathrm{e}^{-\mathrm{i}\theta}$

$h_1\, h_2 = r\left(\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}(\alpha-\theta)} + \mathrm{r}_\theta \mathrm{e}^{\mathrm{i}(\theta-\theta)}\mathrm{p}\right)$

$h_1\, h_2 = r\left(\mathrm{r}_\alpha \mathrm{e}^{\mathrm{i}\varphi} + \mathrm{r}_\theta \mathrm{p}\right)$

Since $|\mathrm{r}_\alpha|^2 + |\mathrm{r}_\theta|^2 = 1$, let $\mathrm{r}_\alpha = \sin\theta$ and $\mathrm{r}_\theta = \cos\theta$. Additionally, $e^{i\varphi} = cos\,\varphi + i.sin\,\varphi$. This gives:

$h_1\, h_2 = r(\sin\theta \times (cos\,\varphi + i.sin\,\varphi) + (\cos\theta)\mathrm{p})$

$h_1\, h_2 = r\sin\theta\, cos\,\varphi + r.i\sin\theta\, sin\,\varphi + r.p\cos\theta$

But substituting Equation (17) and Equation (19) into this result gives:

$h_1\, h_2 = x + yi + zp$

Consequently, for the semi-structured complex number set $\mathbb{H}$, the operation of multiplication is closed, that is $\forall\, h_1 \in \mathbb{H}, \forall\, h_2 \in \mathbb{H},\ h_1\, h_2 \in \mathbb{H}$.

Axiom 7: The operation of multiplication is commutative, that is $\forall\, \boldsymbol{h_1} \in \mathbb{H}, \forall\, \boldsymbol{h_2} \in \mathbb{H},\ \boldsymbol{h_1}\, \boldsymbol{h_2} = \boldsymbol{h_2}\, \boldsymbol{h_1}$.

Consider $h_1 = (x_1 + y_1 i + z_1\mathrm{p}) \in \mathbb{H}$ *and* $h_2 = (x_2 + y_2 i + z_2\mathrm{p}) \in \mathbb{H}$. From axiom 6,

$h_1\, h_2 = (x_1x_2 - y_1y_2 - z_1z_2) + (x_1y_2 + y_1x_2)i + (x_1z_2 + z_1x_2)p + (y_1z_2 + z_1y_2)ip$

Now

$h_2\, h_1 = (x_2 + y_2 i + z_2 p) \times (x_1 + y_1 i + z_1 p)$

$h_2\, h_1 = x_2 \times\ (x_1 + y_1 i + z_1 p) + y_2 i \times\ (x_1 + y_1 i + z_1 p) + z_2 p \times\ (x_1 + y_1 i + z_1 p)$

$h_2\, h_1 = x_2x_1 + x_2y_1 i + x_2z_1 p + y_2x_1 i + y_2y_1 ii + y_2z_1 ip + z_2x_1 p + z_2y_1 ip + z_2z_1 pp$

Now $ii = i^2 = -1$ and $pp = p^2 = p$. Hence

$h_2\, h_1 = x_2x_1 + x_2y_1 i + x_2z_1\mathrm{p} + y_2x_1 i - y_2y_1 + y_2z_1\mathrm{ip} + z_2x_1\mathrm{p} + z_2y_1 ip - z_2z_1$

$h_2\, h_1 = x_2x_1 - z_2z_1 - y_2y_1 + x_2y_1 i + y_2x_1 i + x_2z_1\mathrm{p} + z_2x_1\mathrm{p} + y_2z_1\mathrm{ip} + z_2y_1 ip$

$h_2\, h_1 = (x_1x_2 - y_1y_2 - z_1z_2) + (x_1y_2 + y_1x_2)i + (x_1z_2 + z_1x_2)p + (y_1z_2 + z_1y_2)ip$

It is clear that $h_1\, h_2 = h_2\, h_1$.

Consequently, for the semi-structured complex number set $\mathbb{H}$, the operation of multiplication is commutative, that is $\forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S},\ h_1\, h_2 = h_2\, h_1$.

Axiom 8: The operation of multiplication is associative, that is $\forall \boldsymbol{h_1} \in \mathbb{H}, \forall \boldsymbol{h_2} \in \mathbb{S}, \forall \boldsymbol{h_3} \in \mathbb{H}$, $\boldsymbol{h_1}(\boldsymbol{h_2}\,\boldsymbol{h_3}) = (\boldsymbol{h_1}\,\boldsymbol{h_2})\,\boldsymbol{h_3}$.

Consider $h_1 = (x_1 + y_1 i + z_1\mathrm{p}) \in \mathbb{H}$; $h_2 = (x_2 + y_2 i + z_2\mathrm{p}) \in \mathbb{H}$; $h_3 = (x_3 + y_3 i + z_3\mathrm{p}) \in \mathbb{H}$.

$$h_1(h_2\,h_3) = (x_1 + y_1 i + z_1\mathrm{p}) \times [(x_2 + y_2 i + z_2\mathrm{p}) \times (x_3 + y_3 i + z_3\mathrm{p})]$$

$$h_1(h_2\,h_3) = (x_1 + y_1 i + z_1\mathrm{p}) \times [(x_3x_2 - y_3y_2 - z_3z_2) + (x_3y_2 + y_3x_2)i + (x_3z_2 + z_3x_2)p + (y_3z_2 + z_3y_2)ip]$$

$$\begin{aligned} h_1(h_2\,h_3) = {} & [x_1.(x_3x_2 - y_3y_2 - z_3z_2) - y_1.(x_3y_2 + y_3x_2) - z_1.(x_3z_2 + z_3x_2)] \\ & + [x_1.(x_3y_2 + y_3x_2) + y_1.(x_3x_2 - y_3y_2 - z_3z_2) - z_1.(y_3z_2 + z_3y_2)]i \\ & + [x_1.(x_3z_2 + z_3x_2) - y_1.(y_3z_2 + z_3y_2) + z_1.(x_3x_2 - y_3y_2 - z_3z_2)]p \\ & + [x_1.(y_3z_2 + z_3y_2) + y_1.(x_3z_2 + z_3x_2) + z_1.(x_3y_2 + y_3x_2)]ip \end{aligned}$$

Now consider:

$$(h_1\,h_2)\,h_3 = [(x_1x_2 - y_1y_2 - z_1z_2) + (x_1y_2 + y_1x_2)i + (x_1z_2 + z_1x_2)p + (y_1z_2 + z_1y_2)ip] \times (x_3 + y_3 i + u_3 p)$$

$$\begin{aligned} (h_1\,h_2)\,h_3 = {} & [x_3.(x_1x_2 - y_1y_2 - z_1z_2) - y_3.(x_1y_2 + y_1x_2) - z_3.(x_1z_2 + z_1x_2)] \\ & + [x_3.(x_1y_2 + y_1x_2) + y_3.(x_1x_2 - y_1y_2 - z_1z_2) - z_3.(y_1z_2 + z_1y_2)]i \\ & + [x_3.(x_1z_2 + z_1x_2) - y_3.(y_1z_2 + z_1y_2) + z_3.(x_1x_2 - y_1y_2 - z_1z_2)]p \\ & + [x_3.(y_1z_2 + z_1y_2) + y_3.(x_1z_2 + z_1x_2) + z_3.(x_1y_2 + y_1x_2)]ip \end{aligned}$$

With a bit more calculation it can be seen:

$$\begin{aligned} & [x_3.(x_1x_2 - y_1y_2 - z_1z_2) - y_3.(x_1y_2 + y_1x_2) - z_3.(x_1z_2 + z_1x_2)] \\ & \quad = [x_1.(x_3x_2 - y_3y_2 - z_3z_2) - y_1.(x_3y_2 + y_3x_2) - z_1.(x_3z_2 + z_3x_2)] \\ & [x_3.(x_1y_2 + y_1x_2) + y_3.(x_1x_2 - y_1y_2 - z_1z_2) - z_3.(y_1z_2 + z_1y_2)]i \\ & \quad = [x_1.(x_3y_2 + y_3x_2) + y_1.(x_3x_2 - y_3y_2 - z_3z_2) - z_1.(y_3z_2 + z_3y_2)]i \\ & [x_3.(x_1z_2 + z_1x_2) - y_3.(y_1z_2 + z_1y_2) + z_3.(x_1x_2 - y_1y_2 - z_1z_2)]p \\ & \quad = [x_1.(x_3z_2 + z_3x_2) - y_1.(y_3z_2 + z_3y_2) + z_1.(x_3x_2 - y_3y_2 - z_3z_2)]p \\ & [x_3.(y_1z_2 + z_1y_2) + y_3.(x_1z_2 + z_1x_2) + z_3.(x_1y_2 + y_1x_2)]ip \\ & \quad = [x_1.(y_3z_2 + z_3y_2) + y_1.(x_3z_2 + z_3x_2) + z_1.(x_3y_2 + y_3x_2)]ip \end{aligned}$$

It is clear that $h_1(h_2\,h_3) = (h_1\,h_2)\,h_3$. Consequently, for the semi-structured complex number set $\mathbb{H}$, the operation of multiplication is associative, that is $\forall h_1 \in \mathbb{H}, \forall h_2 \in \mathbb{H}, \forall h_3 \in \mathbb{H}$, $h_1(h_2\,h_3) = (h_1\,h_2)\,h_3$.

Axiom 9: The operation of multiplication has the multiplicative identity element of 1 such that $\forall \boldsymbol{h} \in \mathbb{H}, \mathbf{1}.\boldsymbol{h} = \boldsymbol{h}$.

Consider $h = (x + yi + z\mathrm{p}) \in \mathbb{H}$ *and* $1 = (1 + 0i + 0^2\mathrm{p}) \in \mathbb{H}$.

$$1.h = (1 + 0i + 0^2\mathrm{p}) \times (x + yi + z\mathrm{p})$$

Now from Axiom 6.

$$1.h = (1.x - 0.y - 0^2.z) + (1.y + x.0)i + (1.z + 0^2.x)p + (0^2.y + 0.x)ip$$

$$1.h = (1.x) + (1.y)i + (1.z)p + (0^2)ip$$

$1.h = h$

Consequently, the semi-structured complex number set $\mathbb{H}$, has the multiplicative identity element of 1 such that $\forall h \in \mathbb{H}, 1.h = h$.

Axiom 10: The operation of multiplication has the multiplicative inverse element of $\frac{1}{h}$ such that $\forall h \in \mathbb{S}, h \cdot \frac{1}{h} = 1$.

It needs to be shown that for $h = (x + yi + zp) \in \mathbb{H}$ there exist a $\frac{1}{h} = (q + ri + sp) \in \mathbb{H}$ such that $h \cdot \frac{1}{h} = 1$. From Axiom 6,

$$h \cdot \frac{1}{h} = (xq - yr - zs) + (xr + yq)i + (xs + zq)p + (ys + zr)ip$$

Now suppose $h \cdot \frac{1}{h} = 1$, this implies:

$$1 + 0.i + 0.p = (xq - yr - zs) + (xr + yq)i + (xs + zq)p + (ys + zr)ip$$

If two semi-structured complex numbers are equal, then their real parts, imaginary parts and their unstructured parts must be equal. This results in the following simultaneous equations:

$$xq - yr - zs = 1 \qquad xr + yq = 0 \qquad xs + zq = 0 \qquad ys + zr = 0$$

The solution to these simultaneous equations gives:

$$q = \frac{x}{x^2+y^2+z^2} \qquad r = \frac{-y}{x^2+y^2+z^2} \qquad s = \frac{-z}{x^2+y^2+z^2}$$

So, the reciprocal of $x = (a + bi + cj)$ is the number $\frac{1}{x} = (q + ri + sp)$ where q, r and s have the values just found. In summary, we have the following reciprocation formula:

$$\frac{1}{(x + yi + zp)} = \left(\frac{x}{x^2 + y^2 + z^2}\right) + \left(\frac{-y}{x^2 + y^2 + z^2}\right)i + \left(\frac{-z}{x^2 + y^2 + z^2}\right)p$$

Consequently, for the semi-structured complex number set $\mathbb{H}$, the operation of multiplication has the multiplicative inverse element of $\frac{1}{h}$ such that $\forall h \in \mathbb{H}, h \cdot \frac{1}{h} = 1$.

Axiom 11: The operation of multiplication is distributive over addition, that *is* $\forall h_1 \in \mathbb{H}, \forall h_2 \in \mathbb{H}, \forall h_3 \in \mathbb{H}$, $h_1 \times (h_2 + h_3) = h_1 h_2 + h_1 h_3$.

Consider $h_1 = (x_1 + y_1 i + z_1 \mathrm{p}) \in \mathbb{H}$; $h_2 = (x_2 + y_2 i + z_2 \mathrm{p}) \in \mathbb{H}$ and $h_3 = (x_3 + y_3 i + z_3 \mathrm{p}) \in \mathbb{H}$.

$$h_1 \times (h_2 + h_3) = (x_1 + y_1 i + z_1 \mathrm{p}) \times [(x_2 + y_2 i + z_2 \mathrm{p}) + (x_3 + y_3 i + z_3 \mathrm{p})]$$

$$h_1 \times (h_2 + h_3) = (x_1 + y_1 i + z_1 \mathrm{p}) \times [(x_2 + x_3) + (y_2 + y_3)i + (z_2 + z_3)\mathrm{p}]$$

$$\begin{aligned} h_1 \times (h_2 + h_3) = {} & x_1 \times [(x_2 + x_3) + (y_2 + y_3)i + (z_2 + z_3)\mathrm{p}] \\ & + y_1 i \times [(x_2 + x_3) + (y_2 + y_3)i + (z_2 + z_3)\mathrm{p}] \\ & + u_1 \mathrm{p} \times [(x_2 + x_3) + (y_2 + y_3)i + (z_2 + z_3)\mathrm{p}] \end{aligned}$$

$$\begin{aligned} h_1 \times (h_2 + h_3) = {} & [x_1(x_2 + x_3) + x_1(y_2 + y_3)i + x_1(z_2 + z_3)\mathrm{p}] \\ & + [y_1(x_2 + x_3)i - y_1(y_2 + y_3) + y_1(z_2 + z_3)i\mathrm{p}] \\ & + [z_1(x_2 + x_3)\mathrm{p} + z_1(y_2 + y_3)ip - z_1(z_2 + z_3)] \end{aligned}$$

$$h_1 \times (h_2 + h_3) = [x_1(x_2 + x_3) - y_1(y_2 + y_3) - z_1(z_2 + z_3)] + [x_1(y_2 + y_3) + y_1(x_2 + x_3)]i + [x_1(z_2 + z_3) + z_1(x_2 + x_3)]p + [y_1(z_2 + z_3) + z_1(y_2 + y_3)]ip$$

Now from Axiom 6

$$h_1\, h_2 + h_1\, h_3 = [(x_1x_2 - y_1y_2 - z_1z_2) + (x_1y_2 + y_1x_2)i + (x_1z_2 + z_1x_2)p + (y_1z_2 + z_1y_2)ip] + [(x_1x_3 - y_1y_3 - z_1z_3) + (x_1y_3 + y_1x_3)i + (x_1z_3 + z_1x_3)p + (y_1z_3 + z_1y_3)ip]$$

$$h_1\, h_2 + h_1\, h_3 = [(x_1x_2 - y_1y_2 - z_1z_2 + x_1x_3 - y_1y_3 - z_1z_3) + (x_1y_2 + y_1x_2 + x_1y_3 + y_1x_3)i + (x_1z_2 + z_1x_2 + x_1z_3 + z_1x_3)p + (y_1z_2 + z_1y_2 + y_1z_3 + z_1y_3)ip]$$

With a bit more calculation it is clear that:

$$x_1(x_2 + x_3) - y_1(y_2 + y_3) - z_1(z_2 + z_3) = (x_1x_2 - y_1y_2 - z_1z_2 + x_1x_3 - y_1y_3 - z_1z_3)$$

$$[x_1(y_2 + y_3) + y_1(x_2 + x_3)]i = (x_1y_2 + y_1x_2 + x_1y_3 + y_1x_3)i$$

$$[x_1(z_2 + z_3) + z_1(x_2 + x_3)]p = (x_1z_2 + z_1x_2 + x_1z_3 + z_1x_3)p$$

$$[y_1(z_2 + z_3) + z_1(y_2 + y_3)]ip = (y_1z_2 + z_1y_2 + y_1z_3 + z_1y_3)ip$$

Since the coefficients of $h_1 \times (h_2 + h_3)$ and $h_1\, h_2 + h_1\, h_3$, are the same, this implies that the operation of multiplication is distributive over addition, that $is\ \forall\, h_1 \in \mathbb{H}, \forall\, h_2 \in \mathbb{H}, \forall\, h_3 \in \mathbb{H},\ h_1 \times (h_2 + h_3) = h_1\, h_2 + h_1\, h_3$

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

Peter Jean Paul conducted the research and wrote the paper; Shanaz Wahid proofread the paper correcting grammatical errors and checking for mathematical consistency. All authors have approved the final version.

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- Peter Jean-Paul, Shanaz Wahid, Unstructured and Semi-structured Complex Numbers: A Solution to Division by Zero, Pure and Applied Mathematics Journal. Vol. 10, No. 2, 2021, pp. 49-61. doi: 10.11648/j.pamj.20211002.12
- Jean‐Paul, P., Lie, T. T., Anderson, T. N., & Vallès, B. (2021). A generalized economic model for optimally selecting forecasted load profiles for measuring demand response in residential energy management system. International Journal of Energy Research, 45(11), 16262-16283.
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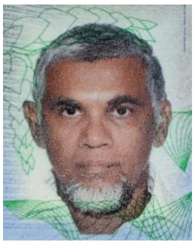

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