Bounded Linear Functional on n-Normed Spaces through Its Quotient Spaces

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Abstract: The aim of this paper is to investigate bounded linear functionals on an n-normed space with respect to norms of its quotient spaces. These norms will be our main tools as well as a new viewpoint. We will give several types of boundedness of bounded linear functionals on an n-normed space and investigate the dual spaces that corresponds to each type of boundedness. We obtain the relations between all types of boundedness, which correspond to the inclusion relations between the dual spaces.

Key words: Bounded linear functionals, n-normed spaces, quotient spaces.

1. Introduction

The concept of n-normed spaces for $n \geq 2$ is a generalization of the concept of normed spaces. This concept was initially introduced by Gahler in 1960's [1]-[4]. Let n be a nonnegative integer and X be a real vector space with $\dim(X) \geq n$. A function $\|\cdot, \dots, \cdot\|: X^n \to \mathbb{R}$ satisfying the following properties

1) $\|x_1, \dots, x_n\| \geq 0$;
$\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent,
2) $\|x_1, \dots, x_n\|$ is invariant under permutation,
3) $\|\alpha x_1, \dots, x_n\| = |\alpha| \, \|x_1, \dots, x_n\|$, for any $\alpha \in \mathbb{R}$,
4) $\|x_1 + x_1', x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x_1', x_2, \dots, x_n\|$,

is called an ***n*-norm** on X, and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an ***n*-normed spaces**. This concept is studied further by many researchers in later years (see for instance [5]-[9]).

Using this concept, we will study bounded linear functionals on an n-normed space. We will use norms of quotient spaces of the n-normed space as our main tools. These norms will be a new viewpoint in investigating some features of n-normed spaces. The new viewpoint is simpler compared to the results obtained by some previous researchers (for more details see [5] and [10]). With respect to these norms, we will define m-bounded linear functionals for each $m \in \{1, \dots, n\}$. We also define other types of boundedness (of a linear functional), which we call p, m-bounded linear functionals, with $p \geq 1$. For $p = 1$, the $1, m$-bounded linear functional are identical to the m-bounded linear functional. Next, we show that for any $p \geq 1$ and a fixed $m \in \{1. \dots, n\}$, all types of p, m-bounded linear functionals are identical. As a consequence, if we compare the dual spaces that correspond to each type of boundedness (with respect to p), they will be identical sets. We also give an inclusion relation between the dual spaces that correspond to boundedness (of a linear functional) with respect to any $m \in \{1, \dots, n\}$. Moreover, instead of using all norms of the quotient spaces to investigate some features of the n-normed space, we show that we can only use some norms of the quotient spaces.

2. Main Results

We shall begin with the construction of quotient spaces of an n-normed space. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space and $Y = \{y_1, \ldots, y_n\}$ is a linearly independent set in X. For a fixed $j \in \{1, \ldots, n\}$, we consider $Y \backslash \{y_j\}$ and then we define the following subspace of X generated by $\backslash \{y_j\}$:

$$Y_j^0 \coloneqq \text{span } Y \backslash \{y_j\} = \left\{ \sum_{i=1, i \neq j}^{n} \alpha_i \, y_i \;\; ; \;\; \alpha_i \in \mathbb{R} \right\}.$$

For any $u \in X$, the corresponding coset of Y_j^0 in X is

$$\bar{u} = \left\{ u + \sum_{i=1, i \neq j}^{n} \alpha_i \, y_i \;\; ; \;\; \alpha_i \in \mathbb{R} \right\}.$$

Hence we have $\bar{0} = \text{span } Y \backslash \{y_j\} = Y_j^0$. We define the quotient space of X as $X_j^* = X / Y_j^0 = \{\bar{u} : u \in X\}$. The addition and the scalar multiplication also apply in X_j^*. Next we define a function $\|\cdot\|_j^* : X_j^* \to \mathbb{R}$ by

$$\|\bar{u}\|_j^* = \left\|u, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n\right\|. \tag{1}$$

This function defines a norm on X_j^*. Then we have $\left(X_j^*, \|\cdot\|_j^*\right)$ is a normed space. By using the above construction we get n quotient spaces, each of them has its own norm. We call the collection of $\left(X_j^*, \|\cdot\|_j^*\right)$ for $j = 1, \ldots, n$ **a class-1 collection** [10].

Furthermore, for a fixed $m \in \{1, \ldots, n\}$ we generalize the above construction by examining $Y \backslash \{y_{i_1}, \ldots, y_{i_m}\}$. For a $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ we define the following subspace of X generated by $Y \backslash \{y_{i_1}, \ldots, y_{i_m}\}$:

$$Y_{i_1, \ldots, i_m}^0 \coloneqq \text{span } Y \backslash \{y_{i_1}, \ldots, y_{i_m}\} = \left\{ \sum_{i=1, i \neq i_1, \ldots, i_m}^{n} \alpha_i \, y_i \;\; ; \;\; \alpha_i \in \mathbb{R} \right\}.$$

For any $u \in X$, the corresponding coset of $Y_{i_1, \ldots, i_m}^0$ in X is

$$\bar{u} = \left\{ u + \sum_{i=1, i \neq i_1, \ldots, i_m}^{n} \alpha_i \, y_i \;\; ; \;\; \alpha_i \in \mathbb{R} \right\}.$$

Hence we have $\bar{0} = \text{span } Y \backslash \{y_{i_1}, \ldots, y_{i_m}\} = Y_{i_1, \ldots, i_m}^0$. Next, we define the quotient space of X as $X_{i_1, \ldots, i_m}^* = X / Y_{i_1, \ldots, i_m}^0 \coloneqq \{\bar{u} : u \in X\}$. The addition and the scalar multiplication also apply in this space. Moreover, define a function $\|\cdot\|_{i_1, \ldots, i_m} : X_{i_1, \ldots, i_m}^* \to \mathbb{R}$ by

$$\|\bar{u}\|_{i_1, \ldots, i_m} = \left\|u, y_1, \ldots, y_{i_1 - 1}, y_{i_1 + 1}, \ldots, y_n\right\| + \cdots + \left\|u, y_1, \ldots, y_{i_m - 1}, y_{i_m + 1}, \ldots, y_n\right\|. \tag{2}$$

The right hand of equation (2) is actually a summation of norms defined in (1). Then we can write equation (2) as

$$\|\bar{u}\|_{i_1,\dots,i_m} = \|\bar{u}\|_{i_1} + \cdots + \|\bar{u}\|_{i_m}. \tag{3}$$

Note that using the above construction, we get $\binom{n}{m}$ quotient spaces. We collect these quotient spaces in a set and name it a **class-m collection** [10]. One can see that for an n-normed space, we can construct n class collections.

Moreover, let X be a real vector space. Recall that $f: X \to \mathbb{R}$ is called **a linear functional** on X, if for any $x, x' \in X$ and $\alpha \in \mathbb{R}$ we have $f(x + x') = f(x) + f(x')$ and $f(\alpha x) = \alpha f(x)$. Next, we will define bounded linear functionals on an n-normed space in several ways as follows. From now on, X will be an n-normed space.

2.1. m-Bounded Linear Functional

For a fixed linearly independent set $Y = \{y_1, \dots, y_n\}$ in X and an $m \in \{1, \dots, n\}$, we say that a linear functional f is **m-bounded with respect to the norms of class-m collection** on X if and only if there is a $K > 0$ such that for any $x \in X$ we have

$$|f(x)| \le K \|x\|_{i_1,\dots,i_m}, \tag{4}$$

for every $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. Moreover, let X'_m be a set that contains all m-bounded linear functionals on X. This set forms a vector space. For any $f \in X$, we define a norm in X'_m as follows,

$$\|f\|_m \coloneqq \inf\{K > 0 \ ; (4) \text{ holds}\}. \tag{5}$$

Then we have the following proposition.

Proposition 2.1.1. The norm in (5) is identical with

$$\|f\|_m \coloneqq \sup\left\{|f(x)| : \|\bar{x}\|_{i_1,\dots,i_m} \le 1, \text{for all } \{i_1, \dots, i_m\} \subset \{1, \dots, n\}\right\}.$$

Proof. Let $m \in \{1, \dots, n\}$, $f \in X'_m$ and $\alpha = \inf\{k > 0 \ ; (4) \text{ holds}\}$. For any $x \in X$ and $\epsilon > 0$ we have

$$\left|f\left(\left(\|\bar{x}\|_{i_1,\dots,i_m} + \epsilon\right)^{-1} x\right)\right| \le \|f\|_m \ ; \qquad \text{for all } \{i_1, \dots, i_m\} \subset \{1, \dots, n\}.$$

One can see that $\left\|\left(\|\bar{x}\|_{i_1,\dots,i_m} + \epsilon\right)^{-1} x\right\|_{i_1,\dots,i_m} \le 1$. Then we have,

$$|f(x)| \le \left(\|\bar{x}\|_{i_1,\dots,i_m} + \epsilon\right) \|f\|_m \ ; \qquad \text{for all } \{i_1, \dots, i_m\} \subset \{1, \dots, n\}.$$

As ϵ can be arbitrarily small, we have

$$|f(x)| \le \|f\|_m \ \|\bar{x}\|_{i_1,\dots,i_m} \qquad \text{for all } \{i_1, \dots, i_m\} \subset \{1, \dots, n\}.$$

We conclude that $\alpha \le \|f\|_m$. Conversely, if $|f(x)| \le K \ \|\bar{x}\|_{i_1,\dots,i_m}$, with $\|\bar{x}\|_{i_1,\dots,i_m} \le 1$ for all $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$, then $|f(x)| \le K$. Because it applies to any $x \in X$ then $\|f\|_m \le K$. Therefore $\|f\|_m \le \alpha$. Hence we have $\|f\|_m = \alpha$, which means both norms are identical.

Next we give a relation between an m_1-bounded linear functional and an m_2-bounded linear functional, for any $m_1, m_2 \in \{1, \dots, n\}$ by the following theorem.

Theorem 2.1.2. Let $m_1, m_2 \in \{1, \dots, n\}$ with $m_1 \le m_2$. If a linear functional f is m_1-bounded then f is also m_2-bounded.

Proof. Let f be a linear functional which m_1-bounded. If $m_1 = m_2$, then obviously f is m_2-bounded.

Let $m_1 < m_2$, then we have

$$|f(x)| \leq K\left(\|\overline{x}\|_{i_1} + \cdots + \|\overline{x}\|_{i_{m_1}} + \cdots + \|\overline{x}\|_{i_{m_2}}\right) \qquad \text{for all } \left\{i_1, \ldots, i_{m_2}\right\} \subset \{1, \ldots, n\}.$$

This means that f is m_2-bounded.

2.2. p, m-Bounded Linear Functional

Let $Y = \{y_1, \ldots, y_n\}$ be a linear independent set in X, $m \in \{1, \ldots, n\}$ and $p \geq 1$. We say a linear functional f is **p, m-bounded with respect to the norms of class-m collection** on X if and only if there is a $K > 0$ such that for any $x \in X$ we have

$$|f(x)| \leq K\left(\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p\right)^{\frac{1}{p}}, \tag{6}$$

for every $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$. One can see that for $p = 1$, the functional f is m-bounded. For $p \geq 1$ and $m \in \{1, \ldots, n\}$, clearly the set $X'_{p,m}$ of all linear functionals that are p, m-bounded on X forms a vector space. We define the following norm on $X'_{p,m}$:

$$\|f\|_{p,m} \coloneqq \inf\{K > 0 \ ; (6) \text{ holds}\}. \tag{7}$$

Note that, based on the norm $\|\cdot\|_{p,m}$ we defined above, we also can write (6) as

$$|f(x)| \leq \|f\|_{p,m}\left(\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p\right)^{\frac{1}{p}}. \tag{8}$$

Proposition 2.2.1. The norm in (7) is identical with

$$\|f\|_{p,m} \coloneqq \sup\left\{|f(x)| : \|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p \leq 1, \text{for all } \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}, x \in X\right\}. \tag{9}$$

Proof. Let $m \in \{1, \ldots, n\}$, $f \in X'_m$ and $\alpha = \inf\{K > 0 \ ; (6) \text{ holds}\}$. For any $x \in X$ and $\epsilon > 0$ we have

$$\left|f\left(\left[\left(\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p\right)^{\frac{1}{p}} + \epsilon\right]^{-1} x\right)\right| \leq \|f\|_{p,m} \qquad \text{for all } \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}.$$

One can see that

$$\left\|\left[\left(\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p\right)^{\frac{1}{p}} + \epsilon\right]^{-1} \overline{x}\right\|_{i_1}^p + \cdots + \left\|\left[\left(\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p\right)^{\frac{1}{p}} + \epsilon\right]^{-1} \overline{x}\right\|_{i_m}^p \leq 1,$$

for all $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$. Then we have,

$$|f(x)| \leq \left(\left(\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p\right)^{\frac{1}{p}} + \epsilon\right) \|f\|_{p,m} \qquad \text{for all } \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}.$$

As ϵ can be arbitrarily small, we obtain

$$|f(x)| \leq \|f\|_{p,m} \left(\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p\right)^{\frac{1}{p}} \qquad \text{for all } \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}.$$

We conclude that $\alpha \leq \|f\|_{p,m}$.

Conversely, if $|f(x)| \leq K\left(\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p\right)^{\frac{1}{p}}$, with $\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p \leq 1$, for all $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$, then $|f(x)| \leq K$. Because it applies to any $x \in X$, we obtain $\|f\|_{p,m} \leq K$. Therefore $\|f\|_{1,m} \leq$

α. Hence we have $\|f\|_{p,m} = \alpha$, which means both norms are identical.

Note that for $p = \infty$ the norm in (9) will be

$$\|f\|_{p,m} \coloneqq \sup\{|f(x)| : \max\{\|\overline{x}\|_{i_1}, \dots, \|\overline{x}\|_{i_m}\} \leq 1 \text{ for all } \{i_1, \dots, i_m\} \subset \{1, \dots, n\}, x \in X\}.$$

Theorem 2.2.2. Let $m_1, m_2 \in \{1, \dots, n\}$, with $m_1 \leq m_2$. If a linear functional f is p, m_1-bounded, then f is p, m_2-bounded.

Proof. The proof is analogous with proof of Theorem 2.1.2.

Furthermore, the following theorem shows that all 'types' of boundedness for any $p \geq 1$ are equivalent, which means that for a fixed $m \in \{1, \dots, n\}$ and for any $p \geq 1$, all dual sets $X'_{p,m}$ are identical.

Theorem 2.2.3. For an $m \in \{1, \dots, n\}$ and $p_1, p_2 \geq 1, a$ linear functional f is p_1, m-bounded if and only if it is p_2, m-bounded. In other words $X'_{p_1,m} = X'_{p_2,m}$.

Proof. We prove the theorem by showing the equivalence between m-bounded and is p, m-bounded linear functionals, for any $p \geq 1$. Let $m \in \{1, \dots, n\}$ and f be an m-bounded linear functional. For $p \geq 1$, if $x \in X$ satisfies $\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p \leq 1$, for all $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$, then using Hölder inequality we have $\|\overline{x}\|_{i_1} + \cdots + \|\overline{x}\|_{i_m} \leq m^{1-\frac{1}{p}}$. Hence

$$\left\|\frac{\overline{x}}{m^{1-\frac{1}{p}}}\right\|_{i_1} + \cdots + \left\|\frac{\overline{x}}{m^{1-\frac{1}{p}}}\right\|_{i_m} \leq 1,$$

for all $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. Moreover, we have $\left|f\left(\frac{x}{m^{1-\frac{1}{p}}}\right)\right| \leq \|f\|_m$ or $|f(x)| \leq m^{1-\frac{1}{p}} \|f\|_m$. Therefore, we conclude that $\|f\|_{p,m} \leq m^{1-\frac{1}{p}} \|f\|_m$. Conversely, let f be a p, m-bounded linear functional. If $x \in X$ satisfies $\|x\|_{i_1,\dots,i_m} \leq 1$, for all $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$, then we can write $\|x\|_{i_1} + \cdots + \|x\|_{i_m} \leq 1$ for all $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. This means that each term is also less than or equal to 1. Hence, for $p \geq 1$, we have $\|x\|_j^p \leq \|x\|_j$ for all $j \in \{1, \dots, n\}$. Therefore we have

$$\|\overline{x}\|_{i_1}^p + \cdots + \|\overline{x}\|_{i_m}^p \leq \|\overline{x}\|_{i_1} + \cdots + \|\overline{x}\|_{i_m} = \|\overline{x}\|_{i_1,\dots,i_m} \leq 1.$$

Consequently $|f(x)| \leq \|f\|_{p,m}$, which means that f is p, m-bounded with $\|f\|_m \leq \|f\|_{p,m}$. We conclude that $\|f\|_m = \|f\|_{p,m}$.

Remark 2.2.4 From the above theorem we also have the equivalence between $\|\cdot\|_m$ and $\|\cdot\|_{p,m}$, that is

$$\|f\|_m \leq \|f\|_{p,m} \leq m^{1-\frac{1}{p}} \|f\|_m.$$

Moreover $X'_{1,m}$ and $X'_{p,m}$ are identical. As a consequence for any $p_1, p_2 \geq 1$, the sets $X'_{p_1,m}$ and $X'_{p_2,m}$ are identical. We will simply say 'm-bounded' instead of 'p, m-bounded'. Since $X'_{p_1,m}$ and $X'_{p_2,m}$ are identical, we also denote it with X'_m unless we need to specify the type.

Moreover, for any $m \in \{1, \dots, n\}$ we also can compare the dual spaces with respect to class-m collection. We write it as a corollary of Theorem 2.2.3 and Theorem 2.2.4.

Corollary 2.2.5 Let $m_1, m_2 \in \{1, \dots, n\}$ with $m_1 \leq m_2$. If a linear functional f is m_1-bounded, then f is m_2-bounded. In other words $X'_{m_1} \subseteq X'_{m_2}$.

Moreover, we can investigate the linear functional of n-normed spaces using some norms $\|\cdot\|_{i_1,\dots,i_m}^*$ of a class-m collection, We just need to choose norms $\|\cdot\|_{i_1,\dots,i_m}^*$ of the class-m collection such that

$$\bigcup\{i_1, \dots, i_m\} \supseteq \{1, \dots, n\}.$$

Furthermore, the least number of norms of a class-m collection that we can choose to investigate the characteristics of the n-normed space is $\left\lceil\frac{m}{n}\right\rceil$ norms.

Conflict of Interest

The authors declare that there is no conflict of interest.

Author Contributions

H.B and H.G conceived the presented idea and developed the theory. H.B provided the proof outline and H.G supervised the results of the work. Both authors discussed the results and contributed to the final manuscript.

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