A Comparison of Fractional and Polynomial Models: Modelling on Number of Subscribers in the Turkish Mobile Telecommunications Market

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Abstract: In this study, a total number of subscribers of Mobile Network Operators (MNOs) in the Turkish mobile telecommunications market modeled mathematically with the fractional approach and polynomial models. The dataset contains the total number of subscribers of MNOs in the Turkish mobile telecommunication market. It consists of annual data between the years of 2004 and 2018. MNOs in the Turkish mobile telecommunications market consists of three company that is Turkcell, Turk Telekom, and Vodafone. The results of the models compared with each other and we obtained the Fractional Model is more successful. Fractional Model provides the opportunity to better modeling of the time series.

Key words: Fractional calculus, polynomial model, telecommunication market.

1. Introduction

Fractional integral and derivative can be defined as the derivative and integral with non-integer order. For the last decades, many scientists and researchers utilized the fractional calculus approach in their studies. Especially, the fractional derivative gives a very good interpretation for the memory, diffusion, and hereditary of a process. The fractional calculation is widely used in diffusion within the lossy media problems, finance, electromagnetic, mathematical modeling of biology, control theory, mechanics [1]-[10].

Nowadays, we have observed a rapid increase and development in mobile communication services. These services utilize industry 4.0, internet of the things (IoT), artificial intelligence and robotics topics which need a well-spread, generalized and developed wide band mobile network. In order to actualize this requirement, it is needed a fair and competitive environment. Both on Earth and in Turkey, official governmental regulation authorities specialized in the telecommunication sector are formed in order to ensure a competitive environment which is the one of the fundamental aims of the authorities. The fact that the regulatory authorities are able to predict what the regulations will develop or what they may cause may be significant and important in terms of conducting the impact analysis of the regulation [11]. Therefore, the mathematical modeling of different parameters and elements of the mobile communication sector takes an important role. In the framework of this study, we investigate the mathematical modelling of the subscriber's number of mobile service providers in Turkey by using fractional calculus approach.

2. Method

In this study, the aim is to compare the performance measurement of the fractional model and polynomial model. The Fractional Model is given as below.

Firstly, the fractional derivative $\mathfrak{D}_x^\alpha$ will be determined from the Riemann-Liouville definition [12] which has the form;

$$\mathfrak{D}_x^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} \frac{f(t)}{(x-t)^\alpha} dt \tag{1}$$

In Equation 1, $\Gamma(1-\alpha)$ stands for Gamma Function which is defined as $\Gamma(1-\alpha) = \int_0^\infty t^{-\alpha} e^{-t} dt$. Here, the Fractional Order (FO), α varies from 0 to 1.

In order to improve the convergence of the Polynomial Model results, we utilized the theory of fractional calculus [12], [13]. In this paper, we assume that the fractional derivative of f(x) is equal to the expression given in Equation 2 in which the derivative order is α and $\alpha \epsilon$ (0, 1).

$$\mathfrak{D}_x^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha} = \sum_{n=1}^{\infty} a_n n x^{n-1} \tag{2}$$

Here, $f(x)$ corresponds to the data of subscriber of mobile service providers with respect time which is denoted as x in the Equation 2.

After, Laplace transform of Equation 2 is taken [11].

$$\mathcal{L}\{f(x)\} = F(s) = \frac{f(0)}{s} + \sum_{n=1}^{\infty} \frac{a_n}{s^{\alpha+n}} \Gamma(n+1) \tag{3}$$

$\mathcal{L}$ corresponds to the Laplace Transform and $\mathcal{L}^{-1}$ stands for the inverse Laplace transform. Laplace transform of $f(x)$ is denoted as $F(s)$. Inverse Laplace transform of Equation 3 is given as

$$\mathcal{L}^{-1}\{F(s)\} = f(x) = f(0) + \sum_{n=1}^{\infty} \frac{a_n \Gamma(n+1) x^{\alpha+n-1}}{\Gamma(\alpha+n)} \tag{4}$$

As mentioned in the introduction part, our purpose is to model a total number of subscribers of Mobile Network Operators (MNOs) in the Turkish mobile telecommunications depending on time by using previously found data, and here, the least square mean method is used [11], [14]-[16]. Due to having the finite number of discrete data for the specific time intervals, also summation corresponds to $f(x)$ in Equation 3 also needs to be truncated to N this yields to have Equation 5 instead of Equation 4.

$$f(x) \cong f(0) + \sum_{n=1}^{N} \frac{a_n \Gamma(n+1) x^{\alpha+n-1}}{\Gamma(\alpha+n)} \tag{5}$$

We have a dataset to make regression on it. The data set is used to make regression with two methods which are the polynomial model and the fractional model offered in this paper.

$$[P_i \; and \; x_i; \; i = 0,1,2,\dots,k]$$

where, x_i represents the time, and P_i corresponds a total number of subscribers of Mobile Network Operators in time intervals. The number of the data in the dataset is K+1. P_i stands for the total number of subscribers of Mobile Network Operators in the specific time given as x_i. The upper limit of N value given in Equation 5 is determined by the dataset dimension.

The square of the error $(\epsilon_i)^2$is defined as the square of the difference between the value P_i and $f(x_i)$

and given in Equation 6. Our purpose is to minimize the square of the total error contributing from the summation of the difference between each value of P_i and corresponding $f(x_i)$ in the least squares method.

$$(\epsilon_i)^2 = \left(P_i - f(x_i)\right)^2 \tag{6}$$

In Equation 7, the summation of error's square is given.

$$\epsilon_T^2 = \sum_{i=0}^{K}(\epsilon_i)^2 \tag{7}$$

By using Equation 5 and 7, Equation 8 is achieved.

$$\epsilon_T^2 = \sum_{i=0}^{K}\left[p_i - \left\{f(0) + \sum_{n=1}^{N}\frac{a_n\Gamma(n+1)x_i^{\alpha+n-1}}{\Gamma(\alpha+n)}\right\}\right]^2 \tag{8}$$

In order to minimize the total error, Equation Set 9 needs to be satisfied [14].

$$\frac{\partial\epsilon_T^2}{\partial f(0)} = 0, \frac{\partial\epsilon_T^2}{\partial a_1} = 0, \frac{\partial\epsilon_T^2}{\partial a_2} = 0, \ldots \frac{\partial\epsilon_T^2}{\partial a_K} = 0 \tag{9}$$

After having Equation Set 9, following System of Linear Algebraic Equations (SLAE) is achieved. SLAE can be denoted as

$$[A]_{N+1\text{x}N+1}[\Omega]_{N+1\text{x}1} = [B]_{N+1\text{x}1} \tag{10}$$

where,

$$[\Omega] = [f(0) \quad a_1 \quad a_2 \quad \ldots \quad a_N]^T$$

$$[\mathrm{B}] = \left[\sum_{i=0}^{K} P_i \quad \sum_{i=0}^{K} P_i x_i^{\alpha} \quad \sum_{i=0}^{K} P_i x_i^{\alpha+1} \quad \ldots \quad \sum_{i=0}^{K} P_i x_i^{\alpha+n-1}\right]^T$$

$$A = \begin{bmatrix} K+1 & \frac{1}{\Gamma(\alpha+1)}\sum_{i=0}^{K} x_i^{\alpha} & \frac{2.1!}{\Gamma(\alpha+2)}\sum_{i=0}^{K} x_i^{\alpha+1} & \cdots & \frac{n!}{\Gamma(\alpha+n)}\sum_{i=0}^{K} x_i^{\alpha+n-1} \\ \sum_{i=0}^{K} x_i^{\alpha} & \frac{1}{\Gamma(\alpha+1)}\sum_{i=0}^{K} x_i^{2\alpha} & \frac{2.1!}{\Gamma(\alpha+2)}\sum_{i=0}^{K} x_i^{2\alpha+1} & \cdots & \frac{n!}{\Gamma(\alpha+n)}\sum_{i=0}^{K} x_i^{2\alpha+n-1} \\ \sum_{i=0}^{K} x_i^{\alpha+1} & \frac{1}{\Gamma(\alpha+1)}\sum_{i=0}^{K} x_i^{2\alpha+1} & \frac{2.1!}{\Gamma(\alpha+2)}\sum_{i=0}^{K} x_i^{2\alpha+2} & \cdots & \frac{n!}{\Gamma(\alpha+n)}\sum_{i=0}^{K} x_i^{2\alpha+n} \\ \vdots & \vdots & \vdots & & \vdots \quad \vdots \\ \sum_{i=0}^{K} x_i^{\alpha+n-1} & \frac{1}{\Gamma(\alpha+1)}\sum_{i=0}^{K} x_i^{2\alpha+n-1} & \frac{2.1!}{\Gamma(\alpha+2)}\sum_{i=0}^{K} x_i^{2\alpha+n} & \cdots & \frac{n!}{\Gamma(\alpha+n)}\sum_{i=0}^{K} x_i^{2(\alpha+n-1)} \end{bmatrix}$$

Here, T stands for the matrix transpose. Unknown coefficients of SLAE in the vector Ω is found by Equation 11.

$$[\Omega]_{N+1x1} = [\mathrm{A}]^{-1}_{N+1xN+1}[B]_{N+1x1} \tag{11}$$

where, $[\mathrm{A}]^{-1}$ is the inverse of [A] matrix.

3. Dataset

In this study, the dataset contains the annual total number of subscribers of MNOs that are Turkcell, Turk Telekom, and Vodafone in the Turkish mobile telecommunication market between 2004 and 2018. The dataset collected from market data report of Turkey Information and Communication Technologies Authority [15]. Fig.1 represents the number of subscribers of total and MNOs in Turkey.

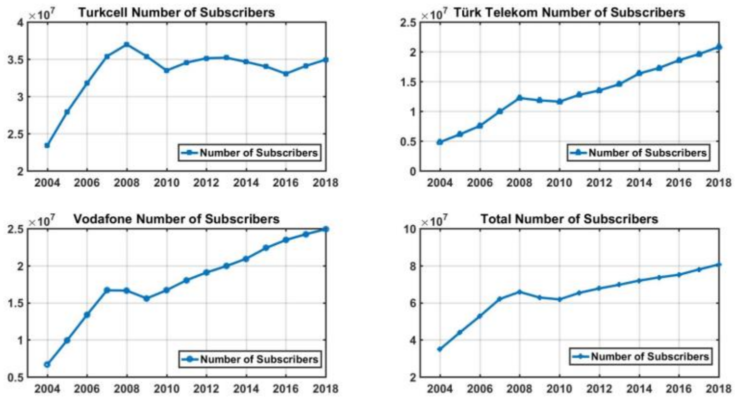

Fig. 1. Number of subscribers of total and mobile network operators.

Fig. 2 represents the number of subscribers of total and MNOs in Turkey. Turkey total number of subscribers is 80.117.999 in 2018. Turkey total number of subscribers increased by 3% from 2017 to 2018. As seen in Fig. 1. Turkcell always obtains market domination. The market share according to the number of subscribers is in Vodafone 31%, Turkcell 42,1%, and Turk Telekom 26,9% for 2018. Also, the distribution of the total revenues is in Vodafone 35,6%, Turkcell 42,4%, and Turk Telekom 22% for 2018 [16].

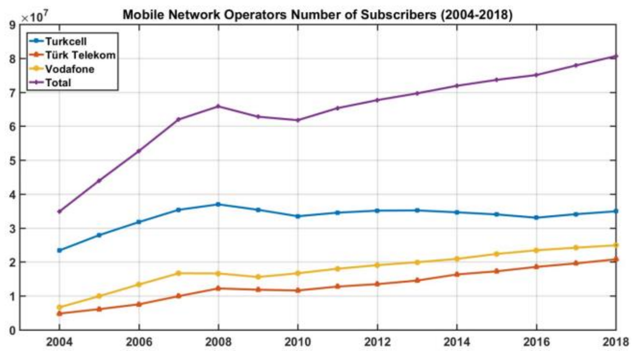

Fig. 2. Number of subscribers of total and MNOs in Turkey.

4. Results

The study aims to compare the performance measurement of the polynomial model and developed a fractional model. This work scope, mathematical modeling of the total number of subscribers of MNOs in the Turkish mobile telecommunication market tried to obtain with three different mathematical models. Truncation number N are chosen 5 and 6 for the models' comparison. The reason of the using different truncation number is better model benchmarking. Matlab 2016b is used for computation. According to the results, Fractional Model generally achieved more successful than Polynomial Model.

Mean Absolute Percentage Error (MAPE) calculate to compare the results of the models. MAPE formulation as seen in (13);

$$MAPE = \frac{1}{k}\sum_{i=1}^{k}\left|\frac{v(i)-\tilde{v}(i)}{v(i)}\right| \times 100 \tag{13}$$

where, $v(i)$ is real number of subscribers value and $\tilde{v}(i)$ is expected value.

Table 1 indicates that MAPE results for the Polynomial Model and Fractional Model. Fractional and Polynomial Models compare with regards to the number of subscribers of MNOs and total in Turkey. According to Table 1, Fractional models generally give better results than Polynomial Model. When Table 1 is analysed, we see that generally lower MAPE values are obtained with Fractional Model. Fractional Model is seen more outdo than Polynomial Model in terms of modeling. Also, when we examine Table 1 as to the truncation number, MAPE gradually decreases when the truncation number is increasing in Fractional Model. Thus, while the truncation number is 6, Fractional Model results and comparison appear in below.

Table 1. MAPE Values of Fractional and Polynomial Model According to the Truncation Number

Truncation Number (*N*)	**Operators**	**Fractional Model**	**Polynomial Model**
		MAPE	**MAPE**
***N*=5**	**Turkcell**	2.157631304	2.172166074
	Turk Telekom	4.487576165	4.487576165
	Vodafone	2.992085434	2.992085434
	Turkey (Total)	2.403936761	2.403936761
***N*=6**	**Turkcell**	2.072421347	2.072421347
	Turk Telekom	2.461999109	3.249563578
	Vodafone	2.709251962	3.054873829
	Turkey (Total)	2.11444334	2.404369749

Fig. 3 indicates that the modeling results of the Turkcell number of subscribers according to Polynomial Model and Fractional Model for *N*=6. When *N* is equal to 6, Polynomial Model and Fractional Model have the same MAPE values because of the alpha value obtained one in the Fractional Model. When the alpha value is one, Polynomial and Fractional Model is equal as mathematically.

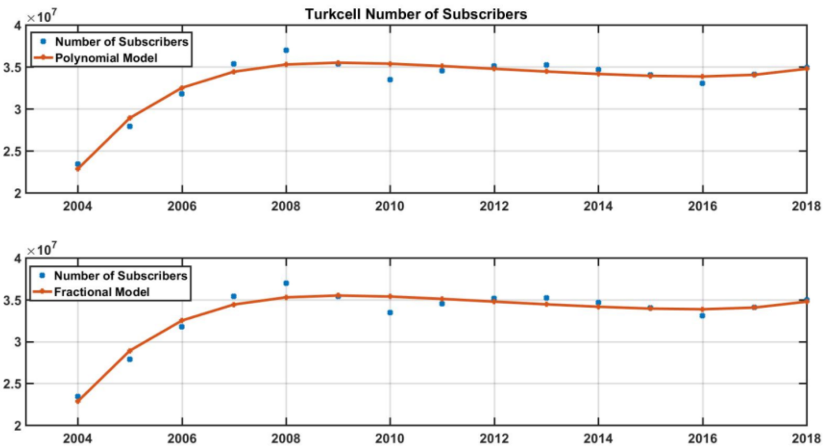

Fig. 3. Modeling results of the Turkcell Number of Subscribers according to polynomial model, fractional model 1, and fractional model 2 for *N*=6.

Fig. 4 indicates that the modeling results of the Turk Telekom Number of Subscribers according to Polynomial Model and Fractional Model for N=6. When N is equal to 6, Polynomial Model is worse 1.32 times than Fractional Model as to the number of subscribers of Turk Telekom.

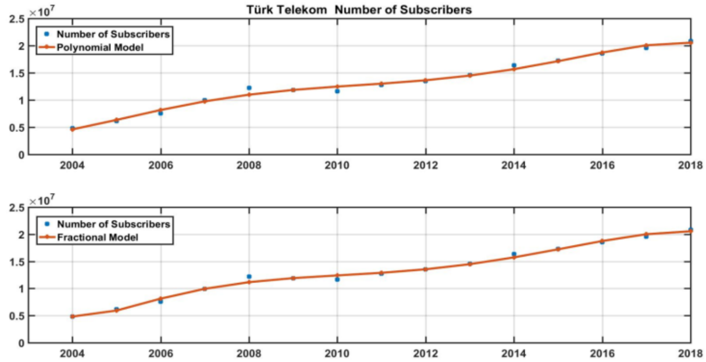

Fig. 4. Modeling results of the Turk Telekom number of subscribers according to polynomial model and fractional model for N=6.

Fig. 5 shows that the modeling results of the Vodafone Number of Subscribers according to Polynomial Model and Fractional Model for N=6. When N is equal to 6, Polynomial Model is worse 1.13 times than Fractional Model as to the number of subscribers of Vodafone.

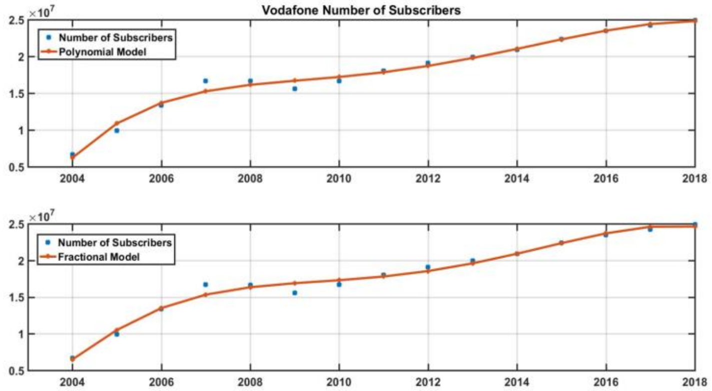

Fig. 5. Modeling results of the vodafone number of subscribers according to polynomial model, fractional model 1, and fractional model 2 for N=6.

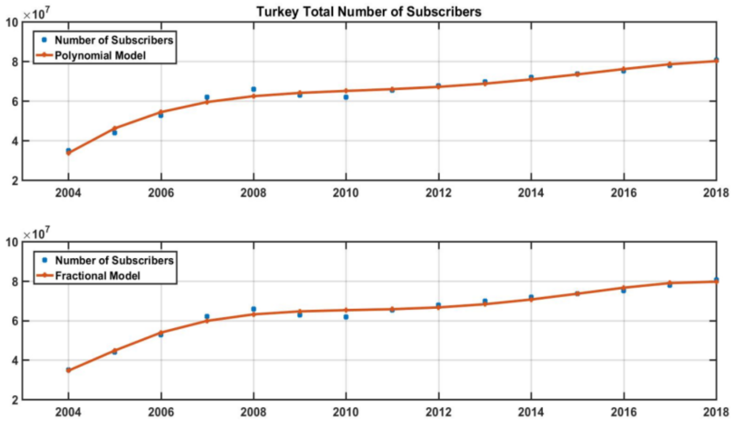

Fig. 6. Modeling results of the Turk Telekom number of subscribers according to polynomial model, fractional model 1, and fractional model 2 for N=6.

Fig. 6 indicates that the modeling results of the Turkey Total Number of Subscribers according to Polynomial Model and Fractional Model for N=6. In order for N=6, MAPE values of Polynomial Model found as 1.14 times worse than Fractional Model for the Turkey total number of subscribers.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Author Contributions

All parts of the paper are writed as team.

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