Justification of Macroscopic Boundary Conditions for One-Dimensional Nonlinear Non-stationary Moment System of Equations of Boltzmann

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Abstract: In this work we prove existence and uniqueness of the solutions of initial and boundary value problem for one-dimensional Boltzmann's moment system of equations with boundary conditions of Maxwell-Auzhan in space of functions continuous in time and summable in square by spatial variable.

Key words: Boltzmann equation, Boltzmann's moment system of equations, Maxwell-Auzhan boundary conditions.

1. Introduction

Aerodynamic characteristics of aircraft at very high speeds and high altitudes can be determined by methods of theory of rarefied gas [1]. For analyzing the aerodynamic characteristics of aircraft in transitional regime the full Boltzmann integro-differential equation is used under appropriate boundary conditions. Determination of boundary conditions on surfaces streamlined by rarefied gas is one of the most important questions of the kinetic theory of gases. In high-altitude aerodynamics an important role is played by interaction of gas with surface of a streamlined body [2]. The aero-thermodynamic characteristics of bodies in gas flow are determined by transfer of momentum and energy to the surface of the body, that is, the relationship between the velocities and energies of molecules incident on the surface and reflected from it, which is essence of kinetic boundary conditions on the surface. Maxwell's boundary condition in solving specific problems more accurately describes the interaction of gas molecules with surface, one of the approximate methods for solving the initial-boundary value problem for Boltzmann equation is moment method. In case of gas flowing near a solid or inside a region bounded by solid surface the boundary conditions describe the interaction of gas molecules with solid walls. The interaction of gas with solid surface is the source of the drag and lifting force of body in gas flow, as well as the heat transfer between gas and the solid boundary. The boundary conditions which particles distribution function must satisfy on the border of the region (where these particles are moving) are depend on the state of boundary surface, on its temperature and on the degree of its roughness and purity. The most common boundary condition for particle distribution function is Maxwell's model which first was presented by Maxwell in 1879. It assumes that fraction of particles that hits the wall areaccomodated by wall and diffusively reflected by Maxwell's distribution. The remaining fraction of particles arespecularly reflected. From Maxwell's accomodation model we obtain boundary condition for the moments of particles distribution function. For calculating the aerodynamic characteristics of aircraft the Boltzmann's moment equations are used. Boltzmann's moment equations are intermediate between Boltzmann (kinetic theory) and hydrodynamic levels of description of state of rarefied gas and form class of nonlinear partial differential equations. Existence of such class of equations was noticed by Grad in 1949 [3], [4]. He obtained the moment system by expanding the particles distribution function in Hermitte polynomials near the local Maxwell distribution. Grad used cartesian coordinates of velocities. Formulation of boundary conditions for Grad's system is almost impossible because the characteristic equations for various approximations of Grad's hyperbolic system contain unknown parameters like density, temperature and average speed.

Boltzmann equation is an equivalent to infinite system of differential equations relative to the moments of particles distribution function in complete system of eigenfunctions of linearized operator. We limit study by finite system of moment equations as solving infinite system of equations is not possible.

Finite system of moment equations for a specific task with a certain degree of accuracy replaces the Boltzmann equation. It's necessary to replace the boundary conditions for particle distribution function by a number of macroscopic conditions for the moments, i.e. there arises problem of boundary conditions for a finite system of equations that approximate the microscopic boundary conditions for the Boltzmann equation. In work [5] we obtained moment system which differs from Grad's system of equations - we used spherical velocity coordinates and decomposed the distribution function into the series of eigenfunctions of linearized collision operator [1], [6], which is the product of Sonin polynomials and spherical functions. The resulting system of equations which correspond to the partial sum of series and which called the Boltzmann's system of moment equations is a nonlinear hyperbolic system in relation to the moments of the particles distribution function.

In this work we prove existence and uniqueness of the solutions of initial and boundary value problem for one-dimensional Boltzmann's moment system of equations with boundary conditions of Maxwell-Auzhan [7] in space of functions continuous in time and summable in square by spatial variable.

2. Statement of the Problem

Find solution of initial-boundary value problem for a homogeneous one-dimensional Boltzmann equation

$$\frac{\partial f}{\partial t}+|v|\cos\theta\frac{\partial f}{\partial x}=J(f,f), t\in(0,T], x\in(-a,a), v\in R_3^v, \tag{1}$$

$$f|_{t=0}=f^0(x,v), (x,v)\in[-a,a]\times R_3^v \tag{2}$$

$$f^+(t,x,v_1,v_2,v_3)=\beta f^-(t,x,v_1,v_2,-v_3)+(1-\beta)exp\left(-\frac{|v|^2}{2RT_0}\right),$$

$$v_3=|v|\cos\theta, (n,v)=(n,|v|\cos\theta)>0, x=-a \text{ or } x=a, \tag{3}$$

where $f\equiv f(t,x,v)$ is particles distribution function in space of velocity and time; $f^0(x,v)$ is distribution of particles at the initial time (fixed function); $J(f,f)\equiv\int[f(v')f(w')-f(v)f(w)]\,\sigma(\cos x)dwdv$ is nonlinear collision operator, recorded for Maxwell molecules, n is the unit external normal vector of boundary.

Condition (3) is natural boundary condition for Boltzmann equation, which makes it possible to determine the reflected half of the distribution function f, if we know the half corresponding to the falling particles. According to (3) some part of falling particles reflected specularly and other particles are absorbed into the wall and emitted with the Maxwell distribution with corresponding wall temperature T_0.

Formula (3) refers to case of wall at rest; otherwise v must be replaced by, $v - u_0$, $u_0$being the velocity of the wall. β, T_0, $u_0$may vary from point to point and with time [6]. In work [7] we approximated microscopic Maxwell boundary condition (3) and formulated initial and boundary value problem for Boltzmann's moment system of equations in arbitrary approximation (odd and even) with Maxwell-Auzhan boundary conditions.

3. Main Results

We study correctness of the initial and boundary value problem for one-dimensional Boltzmann's moment system of equations (we consider pure specular reflection from boundary $(\beta = 1)$)

$$\frac{\partial u}{\partial t} + A\frac{\partial w}{\partial x} = J_1(u, w)$$

$$\frac{\partial w}{\partial t} + A'\frac{\partial u}{\partial x} = J_2(u, w), t \in (0, T], x \in (-a, a), \tag{4}$$

$$u|_{t=0} = u_0(x), w|_{t=0} = w_0(x), x \in [-a, a], \tag{5}$$

$$(Aw^- + Bu^-)|_{x=-a} = (Aw^+ - Bu^+)|_{x=-a} \qquad t \in [0, T], \tag{6}$$

$$(Aw^- - Bu^-)|_{x=a} = (Aw^+ + Bu^+)|_{x=a} \qquad t \in [0, T], \tag{7}$$

where

$$A = \frac{1}{\alpha}\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{\sqrt{3}} & \frac{3}{\sqrt{5}} & -\frac{2\sqrt{2}}{\sqrt{15}} \\ -\sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{5}{3}} \end{pmatrix}, B = \frac{1}{\alpha\sqrt{\pi}}\begin{pmatrix} \sqrt{2} & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & 2\sqrt{2} & -1 \\ -\frac{1}{\sqrt{3}} & -1 & 3\sqrt{2} \end{pmatrix}$$

$$J_1(u, w) = (\ \ 0, J_{02}, 0)', J_2(u, w) = (\ \ 0, J_{03}, J_{11})',$$

$$u = (f_{00}, f_{02}, f_{10})', w = (f_{01}, f_{03}, f_{11})',$$

A' is transpose matrix, B is positive defined matrix;

$$J_{02} = (\sigma_2 - \sigma_0)(f_{00}f_{02} - f_{01}^2/\sqrt{3})/2\,,$$

$$J_{03} = \frac{1}{4}(\sigma_3 + 3\sigma_1 - 4\sigma_0)f_{00}f_{03} + \frac{1}{4\sqrt{5}}(2\sigma_1 + \sigma_0 - 3\sigma_3)f_{01}f_{02}\ ,$$

$$J_{11} = (\sigma_1 - \sigma_0)(f_{00}f_{11} + \frac{1}{2}\sqrt{\frac{5}{3}}f_{10}f_{01} - \frac{\sqrt{2}}{\sqrt{15}}f_{01}f_{02})$$

– are the moments of collision integral, where σ_0, σ_1, σ_2,σ_3 are constants.

It is possible using direct calculations to check that

$$det\,A_1 = det\begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix} \neq 0,$$

and matrix A_1 has three positive and same number of negative nonzero eigenvalues. More exactly $-\sqrt{3+\sqrt{6}}, -1, -\sqrt{3-\sqrt{6}}, \sqrt{3-\sqrt{6}}, 1, \sqrt{3+\sqrt{6}}$ are the eigenvalues of matrix A_1. From (4)-(7) follows that number of boundary conditions on left and right sides of interval (-a, a) is equal to the number of positive and negative eigenvalues of matrix A_1.

Thus a system (4) is symmetric hyperbolic nonlinear partial differential equations system. Let's show that J_{02} is sign-non-defined square form. It is easy to check that $f_{00}f_{02} - f_{01}^2/\sqrt{3} = (CU, U)$, where $U = (u, w)'$,

$u_0(x) = (f_{00}^0(x), f_{02}^0(x), f_{10}^0(x))'$, $w_0(x) = (f_{01}^0(x), f_{03}^0(x), f_{11}^0(x))'$ are the given initial vector-functions; w^+, u^+ are the vector moments of falling to boundary particle distribution function; w^-, u^-- are the vector moments of reflecting from boundary particle distribution function.

$$C = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Eigenvalues of matrix C are -1/2, $-1/\sqrt{3}$, 0, 0, 0, 1/2. Therefore J_{02} is sign-non-defined square form. Similarly we can show that J_{03}, J_{11} are also sign-non-defined square forms.

For the problem (4)-(7) following theorem takes place.

Theorem. If $U_0 = (u_0(x), w_0(x)) \in L^2[-a, a]$, then problem (4)-(7) has unique solution in domain $[-a, a] \times [0, T]$, belonging to the space $C([0, T]; L^2[-a, a])$, moreover

$$\|U\|_{C([0,T];L^2[-a,a])} \leq C_1 \|U_0\|_{L^2[-a,a]} \tag{8}$$

where C_1 is constant independent from U and $T \sim O\left(\|U_0\|_{L^2[-a,a]}^{-1}\right)$.

Proof. Let $U_0 \in L^2[-a, a]$. Let's prove estimation (8). We multiple first equation of system (4) by u and second equation by w, and integrate from –a to a:

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a}[(u,u) + (w,w)]dx + \int_{-a}^{a}\left[\left(A\frac{\partial w}{\partial x}, u\right) + \left(A'\frac{\partial u}{\partial x}, w\right)\right]dx = \int_{-a}^{a}[(J_1, u) + (J_2, w)]dx.$$

After integration by parts we receive

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a}[(u,u) + (w,w)]dx + (u^-, Aw^-)|_{x=a} - (u^-, Aw^-)|_{x=-a} = \int_{-a}^{a}[(J_1, u) + (J_2, w)]dx. \tag{9}$$

Taking into account boundary conditions (6)-(7) we rewrite equality (9) in following form

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a}[(u,u) + (w,w)]dx + (Bu^-, u^-)|_{x=a} + (Bu^-, u^-)|_{x=-a} - \left((Aw^+ - Bu^+), u^-\right)\Big|_{x=-a} +$$
$$+((Aw^+ + Bu^+), u^-)|_{x=a} = \int_{-a}^{a}[(J_1(u,w), u) + (J_2(u,w), w)]dx \tag{10}$$

Let's use spherical representation [8] of vector

$U(t,x)=r(t)\omega(t,x)$, where $\omega(t,x)=(\omega_1(t,x),\omega_2(t,x))'$, $r(t)=\|U(t,.)\|_{L^2[-a,a]}$, $\|\omega\|_{L^2[-a,a]}=1$.
Substituting the values $u=r(t)\omega_1(t,x)$, $w=r(t)\omega_2(t,x)$ into (10) we have that

$$\frac{dr}{dt}+rP(t)=r^2Q(t) \tag{11}$$

where

$$P(t)=(B\omega_1{}^-,\omega_1{}^-)|_{x=a}+(B\omega_1{}^-,\omega_1{}^-)|_{x=-a}+$$
$$+[(A\omega_2{}^+,\omega_1{}^-)_{x=a}+(B\omega_1{}^+,\omega_1{}^-)|_{x=a}+(B\omega_1{}^+,\omega_1{}^-)|_{x=-a}-(A\omega_2{}^+,\omega_1{}^-)|_{x=-a}],$$
$$Q(t)=\int_{-a}^{a}[(J_1(\omega_1,\omega_2),\omega_1)+(J_2(\omega_1,\omega_2),\omega_2)]dx.$$

Let's study equation (11) with initial condition

$$r(0)=\|U_0\|=\|U_0\|_{L^2[-a,a]}. \tag{12}$$

Solution of the problem (11)-(12) has following form

$$r(t)=\left\{exp\left(\int_0^t P(\tau)d\tau\right)\left[\frac{1}{\|U_0\|}-\int_0^t Q(\tau)exp\left(-\int_0^\tau P(\xi)d\xi\right)d\tau\right]\right\}^{-1}.$$

If $R(t)\equiv\int_0^t Q(\tau)exp\left(-\int_0^\tau P(\xi)d\xi\right)d\tau\le 0\ \forall t$, then $r(t)$ is bounded for $\forall t\in[0,+\infty)$. Let $R(t)>0$. We denote by T_1 the moment of time at which

$$\frac{1}{\|U_0\|}-\int_0^{T_1} Q(\tau)exp\left(-\int_0^\tau P(\xi)d\xi\right)d\tau=0.$$

Then $r(t)$ is bounded for $\forall t\in[0,T]$, where $T<T_1$, moreover $T_1\sim O\left(\|U_0\|^{-1}\right)$, since integrand $Q(\tau)exp\left(-\int_0^t P(\xi)d\xi\right)$ is bounded. Hence $\forall t\in[0,T]$ takes place a priori estimation (8).

Now we prove the existence of a solution for (4)-(7) with help of Galerkin method. Let us $\{\omega_l(x)\}_{l=1}^{\infty}$ be a basis in space $L^2[-a,a]$, where dimension of vector $\omega_l(x)$ is equal to dimension of vector U. For each m we define an approximate solution U_m of (4)-(7) as follows:

$$U_m=\sum_{j=1}^{m}c_{jm}(t)v_j(x), \tag{13}$$

$$\int_{-a}^{a}\left(\left(\frac{\partial U_m}{\partial t}+A_1\frac{\partial U_m}{\partial x}\right),v_i(x)\right)dx=\int_{-a}^{a}(J(U_m),v_i(x))dx, i=\overline{1,m}, t\in(0,T], \tag{14}$$

$$U_m|_{t=0}=U_{0m}(x), x\in R, \tag{15}$$

$$(Aw_m^-\mp Bu_m^-)|_{x=\mp a}=(Aw_m^+\pm Bu_m^+)|_{x=\pm a} \tag{16}$$

where U_{0m} is the orthogonal projection in L^2 of function U_0 on the subspace, spanned by $v_1,\dots v_m$.

$$J(u_m)=(J_1(u_m,w_m),J_2(u_m,w_m))'.$$

We represent $v_j(x)$ in the form $v_j(x) = \left(v_j^{(1)}, v_j^{(2)}\right)'$, where

$$v_j^{(1)} = \left(v_{j1}, v_{j2}, v_{j3}\right)', v_j^{(2)} = \left(v_{j4}, v_{j5}, v_{j6}\right)'.$$

The coefficient s$c_{jm}(t)$ are determined from the equations

$$\sum_{j=1}^{m}\left\{\frac{dc_{jm}}{dt}\int_{-a}^{a}(v_j, v_i)dx + c_{jm}\left[\left(Bv_j^{-(1)}, v_i^{-(1)}\right)\Big|_{x=a} + \left(Bv_j^{-(1)}, v_i^{-(1)}\right)\Big|_{x=-a} + \right.\right.$$
$$+\left(Bv_i^{-(1)}, v_j^{-(1)}\right)\Big|_{x=a} + \left(Bv_i^{-(1)}, v_j^{-(1)}\right)\Big|_{x=-a} + \left((Av_i^{+(2)} + Bv_i^{+(1)}), v_j^{-(1)}\right)\Big|_{x=a} -$$
$$-\left((Av_i^{+(2)} - Bv_i^{+(1)}), v_j^{-(1)}\right)\Big|_{x=-a} + \left((Av_j^{+(2)} + Bv_j^{+(1)}), v_i^{-(1)}\right)\Big|_{x=a} -$$
$$\left.\left.-\left(Av_j^{+(2)} - Bv_j^{+(1)}, v_i^{-(1)}\right)\Big|_{x=-a} - \int_{-a}^{a}\left(\left(A\frac{\partial v_i^{(2)}}{\partial x}, v_j^{(1)}\right) + \left(A'\frac{\partial v_i^{(1)}}{\partial x}, v_j^{(2)}\right)\right)dx\right\} =$$
$$= \int_{-a}^{a}\left(J\left(\textstyle\sum_{j=1}^{m} c_{jm}v_j\right), v_i\right)dx\,, i = \overline{1, m}, t \in (0, T] \tag{17}$$

$$c_{im}(0) = d_{im},\ \ \mathrm{i} = \overline{1, \mathrm{m}}, \tag{18}$$

where d_{im} is i-th component of U_{0m}.

We multiply (14) by $\mathrm{c_{im}(t)}$ and sum over i from 1 to m:

$$\int_{-a}^{a}\left(\left(\frac{\partial U_m}{\partial t} + A_1\frac{\partial U_m}{\partial x}\right), U_m\right)dx = \int_{-a}^{a}((J(U_m), U_m)dx.$$

With help of above shown arguments now we prove that $\mathrm{r_m(t)}$ is bounded in some time interval $[0, T_m]$, where $U_m(t, x) = r_m(t)\omega_m(t, x)$, $T_m \approx O\left(\|U_{0m}\|^{-1}\right), T_m \geq T\ \forall m$, and

$$\|U_m\|_{C([0,T];L^2[-a,a])} \leq C_2\|U_0\|_{L^2[-a,a]}, \tag{19}$$

where C_2 is constant and independent from m.

Then solvability of system equations (13)-(16) or (17)-(18) follows from estimation (19).

Thus, the sequence $\{\mathrm{U_m}\}$ of approximate solutions of problem (4)-(7) is uniformly bounded in function space $C([0, T]; L^2[-a, a])$. Moreover, homogeneous system of equations $\tau E + \frac{1}{\alpha}A\xi$ with respect to τ, ξ has only trivial solution. Then it follows from results in [9] that $\mathrm{U_m} \to \mathrm{U}$ is week in $C([0, T]; L^2[-a, a])$ and $J(U_{m,}) \to J(U)$ is week in $C([0, T]; L^2[-a, a])$ as $m \to \infty$. Further, it can be shown by standard methods that limit element is a weak solution of the problem (4)-(7).

The theorem is proved.

4. Concluding Remarks

From this theorem follows that solution existence time depends on the norm of initial vector function. If the norm of initial vector function is small value then solution existence time is big value and if the norm of initial vector function is big value then the solution existence time is small value. Therefore we prove the existence and uniqueness of local on time solution of the initial and boundary value problem for one-dimensional Boltzmann's moment system of equations with boundary conditions of Maxwell-Auzhan in space of functions continuous in time and summable in square by spatial variable.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

Sakabekov A. formulated initial-boundary value problem for one-dimensional nonlinear Boltzmann equation with Maxwell's microscopic boundary conditions; proved the theorem of existence and uniqueness of the solution of initial-boundary value problem for one-dimensional Boltzmann's six-moment system of equations with Maxwell-Auzhan's macroscopic boundary conditions.

Auzhani Y. deduced Maxwell-Auzhan's macroscopic boundary conditions, corresponding to Boltzmann's six-moment system of equations; calculated eigenvalues of matrix A_1; proved sign-non-definedness of quadratic form J_{02}.

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